Abstract

Using a novel nonlinear Bayesian likelihood approach that accounts for the effective lower bound on nominal interest rates, we analyze US post-2008 macroeconomic dynamics. Despite the attention received in the literature, the inclusion of financial frictions does not improve the empirical fit of the standard model or its ability to explain the Great Recession. We further illustrate that the common practice of omitting the ELB period in the estimation severely distorts the analysis of the post-2008 economic dynamics. Our methodological contribution includes a solution method for piecewise-linear models that vastly outperforms existing alternative approaches.

Keywords: Effective Lower Bound, Bayesian Estimation, Great Recession, Business Cycles, Nonlinear Likelihood Inference

JEL: C11, C63, E31, E32, E44 C11 C63 E32 E52

1 Introduction

More than a decade ago, the Financial Crisis and the subsequent Great Recession did not only wreak havoc on the US economy, but it also shook the macroeconomic profession to the core. As a consequence, a plethora of approaches have been developed to enrich dynamic macroeconomic models with features conceived to enhance our understanding of the dynamics during and after the Great Recession, in particular, financial frictions. While progress flourished on the front of theoretical modeling, very few attempts have been made to test these models empirically on the period including and following the Great Recession. This is primarily due to the long-lasting binding effective lower bound on nominal interest rates (ELB), which renders conventional econometric methods unsuitable.
In this paper, we make a step towards closing this gap by proposing a fast solution method for piecewise-linear models that, in conjunction with a novel and fully Bayesian non-linear filter, allows us to estimate even large macroeconomic models while accounting for the effects of the ELB. We first employ our approach to estimate the standard medium-scale representative agent new Keynesian model (RANK) of Christiano et al. (2005) and Smets and Wouters (2007) on a sample that extends to 2019. Thereby we are the first that also include the exit from the ELB in our estimation. In this canonical framework, we illustrate the importance of including the observations of the ELB period in the estimation. Then, motivated by the importance of the interlinkages between the financial sector and the real economy during the Great Recession, we consider the extension of the standard framework developed by Del Negro et al. (2015), who add financial frictions inspired by Bernanke et al. (1999). Using the extended model (FRANK), we show that the inclusion of financial frictions does not improve the model’s ability to explain macroeconomic dynamics in the Great Recession.

The inclusion of post-2008 data has highly relevant implications on the business cycle properties of the estimated models. It has become a common practice to analyze the Great Recession and the ELB period through the lens of models that have been calibrated to or estimated on pre-crisis data only (see, e.g., Gertler and Karadi, 2011; Christiano et al., 2014, 2015; Del Negro et al., 2015; Carlstrom et al., 2017). This approach has generated prominent results that shape our profession’s understanding of the Great Recession, the role of financial frictions, and the effect of unconventional monetary policy. We illustrate that this practice can generate misleading conclusions. We compare the decomposition of macroeconomic dynamics derived from the RANK model estimated on a sample, which includes post-crisis data, with a decomposition of these dynamics as implied by using pre-crisis data only. This exercise reveals that the sample choice substantially affects the contribution of the different driving forces in the model. In the full sample, elevated risk premiums in household financing are the dominating driver of the crisis. In contrast, an analysis based on pre-crisis data overstates the importance of shocks to firms’ investment financing. This illustrates why previous model-driven studies that based their empirical analysis on pre-2008 data focused on disturbances to investment financing as a driver of the crisis, rather than shocks to the household sector.

As our second contribution we assess whether extending the model with financial frictions improves upon the standard model’s performance in terms of empirical fit. We document that the simple RANK model clearly outperforms the model with financial frictions. The main difference between the models is that in the latter, the required return on capital is tied to the leverage of entrepreneurs. By construction, a higher leverage raises the credit spread in the model. However, while the observed spread remains elevated after the Great Recession, we show that the model-implied leverage ratio barely increases in the crisis, but falls substantially thereafter. This result

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1. This shock can either be interpreted as an economy-wide increase in the demand for liquid or safe assets (Fisher, 2015), or it can be associated with the importance of household financing for the Great Recession. This latter interpretation analysis corroborates the work by Mian and Sufi (2014, 2015) and is in line with Kehoe et al. (2020), who argue that the credit tightening for households contributed more to the downturn than the credit tightening for firms.

2. By considering the role of financial friction for the US economy in the Great Recession, we touch upon an active literature, see e.g. Meh and Moran (2010); Gerali et al. (2010); Cúrdia and Woodford (2011); Gertler and Karadi (2011); Brunnermeier and Sannikov (2014); Christiano et al. (2014); Del Negro et al. (2017). Our analysis provides an argument for the benefits of testing these models empirically on the period of the Great Recession and the ELB.

3. The dynamics of the leverage-ratio in our model is in line with the dynamics of the leverage of financial institutions in the Great Recession as documented by Adrian and Shin (2014).
is robust to employing shorter samples that have a narrower focus on the Great Recession. In the estimated model this deleveraging can be traced back to the substantial decline of the capital stock, which is driven by the collapse of investment in the Great Recession. The divergence of the spread and its endogenous driver (the leverage ratio) must be compensated by additional financial shocks, which worsens the empirical performance of the model compared to the plain RANK model.

The finding of a lower empirical fit for the model with financial frictions may appear odd, as its superiority for forecasts has been discussed by Del Negro and Schorfheide (2013); Del Negro et al. (2015); Cai et al. (2019). Indeed, the introduction of financial frictions allows for a model-consistent role of financial spreads for macroeconomic dynamics. As financial data is available earlier than national account data and published on a higher frequency, this provides a substantial advantage for nowcasts and forecasts in a financial crisis. Nonetheless, we show that financial frictions do not improve upon the ex-post explanation of the Great Recession.

We show that the introduction of the financial sector attenuates the decline of investment in response to risk premium shocks and shocks to the marginal efficiency of investment. In the case of a recessionary shock of either type, the decline of the leverage ratio has a dampening impact on the required return on investment.\textsuperscript{4} The financial sector therefore reduces the differential between the reductions of consumption and investment in the Great Recession. As the observed drop-differential of investment and consumption was sizable due to the collapse of investment, the financial sector impedes the ability of these shocks to explain the Recession. In addition, financial shocks as in Christiano et al. (2014) only play a minor role for macroeconomic dynamics. Thus, while, in principle, the introduction of financial frictions yields the advantage that it allows to further discipline model estimates with financial data series and opens the door for the analysis of the effects of financial shocks, the relevance of these shocks is low in the estimated model.

Our technical contribution is to develop our novel solution method and combine it with novel econometric techniques, that enable us to estimate medium to large scale models with an endogenously binding ELB quickly and robustly. We first show that the solution of a piecewise-linear dynamic rational expectations system with occasionally binding constraints (OBCs), depending on the expected duration of the constraint, can be represented in closed form. We then develop a set of simple equilibrium conditions that, together with the closed form solutions, can be used to avoid matrix inversions and simulations at runtime for significant gains in computational speed. An efficient implementation is provided in Python programming language. Benchmarking results show that for medium-scale models with an OBC, more than 150,000 state vectors can be evaluated per second. This is an improvement of more than three orders of magnitude over previous approaches to the solution of piece-wise linear models.\textsuperscript{5}

The novel solution method is coupled with the Ensemble Kalman Filter (EnKF) introduced in Evensen (1994) that accounts for the uncertainty in initial states and the potential for measurement errors while preserving the advantage in speed, and the differential evolution ensemble Monte-Carlo Markov chain method (DE-MCMC) (ter Braak, 2006; ter Braak and Vrugt, 2008) as a posterior sampler, which renders the results robust to local maxima in the likelihood function. Despite the methodological differences, our analysis of the baseline RANK model confirms previous findings of Gust et al. (2017) and Kulish et al. (2017), who consider a binding ELB within comparable models. This lends plausibility to our further analysis and suggests that the loss of precision that might

\textsuperscript{4}In the estimated FRANK model, this is counterbalanced by the higher estimated persistence of both shocks relative to the RANK model.

\textsuperscript{5}For comparison (see, e.g., Guerrieri and Iacoviello, 2015; Holden, 2017)
The notion is supported by Atkinson et al. (2019), who compare piecewise-linear solutions to models with OBCs with fully global methods. They conclude that, while the fully nonlinear solution entails some nice properties, the piecewise-linear solution is to be preferred as it enables the use of larger and hence much less misspecified models.

This observation fueled a literature on the *Missing Deflation Puzzle*. See, e.g., Hall (2011), King and Watson (2012).
In this model, entrepreneurs obtain loans from frictionless intermediaries, which in turn receive their funds from household at the riskless interest rate. In addition to the loans, entrepreneurs use their own net worth to finance the purchase of physical capital, that they rent out to intermediate good producers. Entrepreneurs are subject to idiosyncratic shocks to their success in managing capital. As a consequence, their revenue might fall short of the amount needed to repay the loan, in which case they will default on their loan. In anticipation of the risk of entrepreneurs’ default, financial intermediates pool their loans and charge a spread on the riskless rate to cover the expected losses arising from defaulting entrepreneurs. Crucially, the spread of the loan rate \( \tilde{r}_k \) over the risk free nominal interest rate, \( r_t \), depends on the entrepreneurial leverage and can be written as

\[
E_t[\tilde{r}_{k+1}^t - r_t] = u_t + \zeta_{sp,b}(q_t + \bar{K}_t - n_t) + \tilde{\sigma}_{\omega, t}.
\] (2)

Here, \( u_t \) is the risk premium shock on the households borrowing rate, \( q_t \) is the price of capital, \( k_t \) is the capital stock and \( n_t \) denotes entrepreneurial net worth. \( \tilde{\sigma}_{\omega, t} \) is a shock to the entrepreneurs’ riskiness and follows an AR(1) process - the risk shock introduced by Christiano et al. (2014). Thus, the loan spread is defined as a function of the entrepreneurs’ leverage and their riskiness, which is determined by the dispersion of the idiosyncratic shocks to entrepreneurs. The real loan rate is linked to the return on capital by

\[
\tilde{r}_k^t - \pi_t = \frac{r_k}{r_k + (1 - \delta)} r_k^t + \frac{(1 - \delta)}{r_k + (1 - \delta)} q_t - q_{t-1},
\] (3)

where \( \pi_t \) is the inflation rate, \( r_k^t \) and \( r^k \) denote the dynamics and the steady state of the marginal product of capital, and parameter \( \delta \) is the depreciation rate. Note that if the elasticity of the loan rate to the entrepreneurs’ leverage, \( \zeta_{sp,b} \), is set to zero we are back to the case without financial frictions. The left hand side of equation 3 then reads \((r_t - \pi_t + u_t)\) as in the Smets and Wouters (2007) model.

The evolution of aggregate entrepreneurial net worth is described by

\[
n_t = \zeta_{n, \tilde{r}}(\tilde{r}_k^t - \pi_t) - \zeta_{n, r}(q_{t-1} + \bar{K}_{t-1}) + \zeta_{n, qk}(q_t + \bar{K}_t - n_t) - \zeta_{n, \sigma_{\omega}} - \gamma_s \frac{v_s}{n_s} \tilde{z}_t.
\] (4)

Equation (4) links the accumulated stock of entrepreneurial net worth to the real return of renting out capital to firms, the riskless real rate, its capital holdings, its past net worth and variations in riskiness. The technology process enters the equation due to the form of detrending we borrow from Del Negro et al. (2015). The coefficients \( \zeta_{n, \tilde{r}}, \zeta_{n, r}, \zeta_{n, qk}, \zeta_{n, \sigma_{\omega}}, \gamma_s, v_s \) and \( n_s \) are derived as in Del Negro et al. (2015).

3 Methodology and Data

Data samples in which the ELB binds pose a host of technical challenges for the estimation of DSGE models. These are related to the solution, likelihood inference, and posterior sampling of models in the presence of an occasionally binding constraint (OBC). While methods to solve models with OBCs exists, and – likewise – nonlinear filters are available, the combination of both is computationally very expensive for medium-scale models. Hence, very few examples in the literature
were able to follow this approach (Gust et al., 2017; Kulish et al., 2017). In this section, we briefly summarize the set of novel methods that allow us to conduct the estimation of medium-scale models in the presence of an occasionally binding ELB. Secondly, we discuss our choices with regard to the data, calibrated parameters, and priors used in the empirical analysis.

3.1 A method to deal with occasionally binding constraints efficiently

We develop a novel method to solve for the occasionally binding ELB efficiently and robustly. This algorithm shares some features with the method of Occbin (Guerrieri and Iacoviello, 2015), which is currently the most frequently used algorithm for OBCs. While the method presented in the paper at hand shares some properties with that algorithm (and, given uniqueness, will return an identical solution), it has a considerable advantage in terms of computation speed and, hence, is particularly well suited for parameter inference as well as for very high dimensional models. As we document in Appendix A.4, our method performs about 1500 times faster than Guerrieri and Iacoviello (2015). Another method to solve models with OBCs is presented in Holden (2016, 2017). This method is robust and accurate, especially with regard to proper equilibrium selection. It is however not targeted and optimized in terms of computational speed. As the outcomes of the method presented in this paper is, given uniqueness, identical to those of the work cited above, we refer to the papers cited above for comparisons with other nonlinear methods such as policy function iteration.

The high computational cost of solving economic models stems from the modeling of rational expectations. In the context of OBCs, this implies that agents must foresee for how long a constraint binds. In the worst case, this implies trial-and-errors simulations until a consistent rational expectations solution is found. We develop a closed-form state-space representation for the complete expected trajectory of the endogenous variables as a function of the expected duration at the ELB (the “ELB spell duration”). We furthermore provide the necessary conditions of a rational expectations equilibrium for a set of ELB spell durations given the state of the economy. Given these two ingredients, the expected ELB spell durations can be found via a simple iterative scheme. Using the closed-form solution together with the equilibrium conditions allows to check for a model equilibrium instantaneously instead of simulating a complete anticipated equilibrium path for a given ELB spell. This increases the computational speed of the algorithm substantially.

The model is linearized around its steady state balanced growth path. Respecting the ELB then results in a piecewise-linear model. Generically, let the occasionally binding constraint be given by

$$ r_t = \max \{ ay_{t+1} + by_t + cy_{t-1}, \bar{r} \}. $$

Then, inspired by Uhlig et al. (1995), Binder and Pesaran (1995), and Villemot et al. (2011), the linearized first order conditions of our model can be represented as

$$ E_t [ Ay_{t+1} + By_t + Cy_{t-1} + h \otimes \max \{ ay_{t+1} + by_t + cy_{t-1}, \bar{r} \} ] = 0, $$

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8The estimation of DSGE models with a binding ELB was pioneered by work on small-scale NK models. See, e.g., Keen et al. (2017), Borağan Aruoba et al. (2018), Plante et al. (2018).

9The authors propose a recursive representation of the solution given the state of the economy and a set of spell durations. They propose a Newton-like method to iteratively find the set of spell durations. The relatively high computational costs of their approach stems from the fact that for each guess of the spell durations, the Newton-like method requires the complete simulation of the anticipated trajectory. This requires repeated matrix inversions at runtime, which are computationally expensive.
where \( y_t \) contains all endogenous and exogenous variables and \( A, B \) and \( C \) are generic \( n \times n \) system matrices. In Appendix A we give detailed account on how to include exogenous variables into this system. \( \otimes \) denotes the inner product and hence \( h \otimes \max\{\cdot, \bar{r}\} \) can either be a matrix or a column vector, depending on whether the constraint binds or not. Define the system in which the constraints is slack (the unconstrained system) as

\[
\hat{A}y_{t+1} + \hat{B}y_t + \hat{C}y_{t-1} = 0,
\]

with

\[
\hat{Z} = Z + h \otimes z \text{ for } (Z, z) \in \{(A, a), (B, b), (C, c)\}.
\]

This means that e.g. the unconstrained matrix \( \hat{B} \) for contemporary variables \( y_t \) is the matrix \( B \) of the system for the case that the constraint binds plus a correction matrix, which contains the endogenous responses of \( r_t \) if the constraint is slack. We assume that the unconstrained system satisfies all conditions for determinancy, which are given in Appendix A.

To facilitate a solution, Equation (A.3) can be restated as a first-order difference equation (see e.g. Klein (2000) or Villemot et al. (2011)).

\[
P E_t x_{t+1} = M x_t + h \otimes \max\{p E_t x_{t+1} + m x_t, \bar{r}\},
\]

with \( x_t = \begin{bmatrix} s_{t-1} \\ c_t \end{bmatrix} \), where \( c_t \) are the forward looking variables (controls) and \( s_{t-1} \) are the states updated by the time-\( t \) shocks as above. Note that the OBC now reads as \( r_t = \max\{p E_t x_{t+1} + m x_t, \bar{r}\} \).

Analog to equations (A.5)-(A.7), denote the system in which the constraint is slack (the unconstrained system) as

\[
\hat{P} E_t x_{t+1} = \hat{M} x_t,
\]

with

\[
\hat{P} = P + h \otimes p \quad \text{and} \quad \hat{M} = M + h \otimes m,
\]

which are again the system matrices of the constrained system plus a correction matrix for the endogenous responses of \( r_t \).

For this section we will assume that \( P \) and \( \hat{P} \) are invertible as this simplifies display. The details on how to deal with a singular \( P \) or \( \hat{P} \) are given in Appendix A. Under this assumption, system (11) can be rewritten as

\[
E_t x_{t+1} = \begin{cases} \hat{N} x_t & \forall p E_t x_{t+1} + m x_t - \bar{r} \geq 0 \\ N x_t + q \bar{r} & \forall p E_t x_{t+1} + m x_t - \bar{r} < 0, \end{cases}
\]

with \( \hat{N} = \hat{P}^{-1} \hat{M}, N = P^{-1} M \) and \( q = P^{-1} h \). If the constraint is slack, the upper case holds with \( E_t x_{t+1} = \hat{N} x_t \) and the lower case holds, if the constraint binds.

\[\text{The fundamental idea is that}
\]

\[
A y_{t+1} + B y_t + C y_{t-1} = 0 \iff \begin{vmatrix} A & B \\ 0 & I \end{vmatrix} \begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} + \begin{vmatrix} 0 & C \\ -I & 0 \end{vmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = 0.
\]
To fix ideas, we first assume that if the constraint binds, it always binds immediately in the current period \( t \) (hence there is no transition to the constraint). Let \( k_t \) be the expected duration of the spell to the constraint in period \( t \). Denote a rational expectations solution to (11) given \( k_t \) and the state variables \( s_{t-1} \) as the function \( f \) such that

\[
c_t = f(k_t, s_{t-1}).
\] (14)

In slight abuse of notation, we will use \( k \) and \( f(k) \) as shorthand where the states \( s_{t-1} \) and the time-\( t \) subscripts are understood. Also, denote as \( x_t|k \) the variables conditional on expecting the constraint to hold for \( k \) periods, which is trivial to find once \( f \) is known.

For the unconstrained system \( \hat{N} \), the controls \( c_t \) can be found using familiar methods like the QZ-decomposition as suggested by Klein (2000). Denote this (linear) solution for \( c_t \) by the matrix \( \Omega \):

\[
c_t = \Omega s_{t-1} \quad \forall pE_t x_{t+1} + mx_t > \bar{r}
\] (15)

For \( \Psi = [-\Omega \ I] \), Equation (15) implies that

\[
E_t \left\{ \Psi \begin{bmatrix} s_{t+k} \\ c_{t+k+1} \end{bmatrix} \right\} = 0 \quad \forall pE_t x_{t+k+1} + mx_{t+k} \geq \bar{r},
\] (16)

i.e. for every future period \( t + k \) in which the system is expected to be unconstrained.

Now assume that the constraint binds at time \( t \) and will continue to do so until period \( t + k \). Iterating System (13) \( k \) periods forward yields

\[
E_t \left\{ \begin{bmatrix} s_{t+k} \\ c_{t+k} \end{bmatrix} \right\} = N^k x_t + (I - N)^{-1}(I - N^k)q\bar{r},
\] (17)

where \( N^k = NN \ldots N \) \( k \) times denotes the matrix power and \( (I - N)^{-1}(I - N^k) = \sum_{i=0}^{k-1} N^i \) comes from the transformation for a geometric series of matrices. Finally, we can combine Equations (16) and (17) to find \( f \), i.e. a solution of the controls \( c_t \) in terms of the state variables \( s_{t-1} \) given \( k \):

\[
f(k, s_{t-1}) = \left( c_t : \Psi N^k \hat{N}^t \begin{bmatrix} s_{t-1} \\ c_t \end{bmatrix} \right) = -\Psi(I - N)^{-1}(I - N^k)q\bar{r}.
\] (18)

Since \( q \) is a vector of constants, the whole RHS of Equation (18) is known and solving for \( c_t \) is a simple act of linear algebra.

Let us now relax the assumption that a shock triggers the constraint to hold immediately in time \( t \). This case is in particular relevant for models with persistent endogenous state variables. Take Equation (18) as the starting point and allow for a number of periods \( l \) in the unconstrained system \( \hat{N} \) until the system is at the constraint. We can proceed in a similar manner as above to obtain

\[
f(k, l, s_{t-1}) = \left( c_t : \Psi N^k \hat{N}^l \begin{bmatrix} s_{t-1} \\ c_t \end{bmatrix} \right) = -\Psi(I - N)^{-1}(I - N^k)q\bar{r}.
\] (19)

Using Equation (19) and Equation (17) augmented by \( \hat{N}^l \), we can express the expectations on the variable vector conditional on \((l_t, k_t)\) of the economy in period \( j \), \( E_t x_j(l_t, k_t) \), as a function \( F \) with
\[ E_t[x_j](l_t, k_t) = F_j(l_t, k_t, s_{t-1}) = N^{\max(s-t,0)} N^{\min(t,s)} \left| f(l_t, k_t, s_{t-1}) \right| \]
\[ + (I - N)^{-1}(I - N^{\max(j-l,0)})q \bar{r}. \]  

Note that \( F_1(0, 0, j_{t-1}) \) is the generic solution to the unconstrained system.

Equation (20) is the key equation of the algorithm. It states that for any guess on \((l, k)\), the expected value in any period \(s\) of all endogenous variables, \(y_{t+s}\), can be readily obtained by using only the system matrix of the constrained system \(N\), the coefficients \(q\) of the constraint variable, and the solution of the unconstrained system \(\Psi\).

### 3.2 Solving for the spell durations \((l, k)\)

Again, let us first consider the simpler case in which we assume that any shock that causes the constraint to bind, will cause it to bind immediately in time \(t\) (the non-transitory case). The following proposition summarizes the conditions for \((x_t, s_{t-1}, k)\) to be a rational expectations equilibrium:

**Proposition 1** (non-transitory equilibrium). Assuming non-transition, a number of expected periods \(k^*\) at the constraint is part of a rational expectations equilibrium iff

\[ pE_t[x_{t+k+1}|k^*] + mx_{t+k}|k^* \geq \bar{r} > pE_t[x_{t+k+1}|k] + mx_{t+k}|k \]

for all \(k^* > k \geq 0\), hence if in expectations the system is constrained for exactly \(k^*\) periods.

**Proof.** The proof follows from the definition of \(k\) and \(r_t\). First acknowledge that \(E_t r_{t+j} = pE_t x_{t+j+1} + mE_t x_{t+j}\) and hence the expected value of \(r\) in period \(t + j\) conditional on a guess \(k^*\) is given by

\[ E_t[r_{t+j}|k^*] = pE_t[x_{t+j+1}|k^*] + mE_t[x_{t+j}|k^*]. \]  

For \(k^*\) to be an equilibrium it is required that the constraint binds for exactly \(k^*\) periods. This would be violated if \(E_t[r_{t+j}|k^*] > \bar{r}\) for any \(j < k^*\). Similarly, \(k^*\) cannot be an equilibrium if \(E_t[r_{t+j}|k^*] < \bar{r}\) for any \(j > k^*\).

Let us proceed to the practically relevant case where agents may expect the unconstrained system to prevail for some transition time before the constraint binds for \(k\) periods. Using Equation (20), Proposition 2 summarizes the respective equilibrium conditions.

**Proposition 2** (transitory equilibrium). A pair \((l^*, k^*)\) is part of a rational expectations equilibrium iff

\[ pE_t[x_{t+j+1}|(l^*, k^*)] + mx_{t+j}|(l^*, k^*) \geq \bar{r} \quad \forall j < l^* \wedge j \geq k^* + l^* \]

and

\[ pE_t[x_{t+j+1}|(l^*, k^*)] + mx_{t+j}|(l^*, k^*) < \bar{r} \quad \forall l^* \leq j < k^* + l^*. \]

**Proof.** The proof is analogue to the proof of Proposition 1. Additionally, for \(l^*\) to be an equilibrium it is required that the constraint is slack for exactly \(l^*\) periods, before becoming binding in period \(l^*\). This requires that

\[ E_t[r_{t+j}|(l^*, k^*)] > \bar{r} \quad \forall 0 \leq j < l^* \]  

9
since otherwise \( l^* \) cannot be an equilibrium.

In other words, \((l, k)\) are part of an equilibrium, if in expectations, the constraint starts binding exactly in period \( t + l \) and ends to bind exactly in period \( t + l + k \).

Unfortunately, there is no closed form solution for \((l^*, k^*)\) given \(s_{t-1}\). A set of \((l^*, k^*)\) that satisfies Proposition 2 must be found using an iterative scheme. As this problem requires an iterative scheme on an integer domain, a theoretical assessment is at least difficult because most theoretical work on similar algorithms deals with real valued functions. Given those limits, some insights regarding the existence and uniqueness of such solutions are provided by Holden (2017). Note that an integer based Newton-like method as employed by Guerrieri and Iacoviello (2015) is not efficient because it requires the evaluation of the complete anticipated trajectory of \( x_t \) and furthermore can not draw on the nice convergence properties of the standard real-valued Newton method. This renders their method impractical when OBC problems are not well behaved. Note further that for a given \( s_{t-1} \), an equilibrium satisfying conditions (23) and (24) will generally be the same as in Holden (2017).

The crucial advantage of the formulation provided above is the closed form expression of \( E_t[x_{t+j}|(l, k)] \). Given the set of expected periods \((j, l, k)\), the policy function (20) is linear in \( s_{t-1} \) and consists of a pair \((z_1(j, l, k), z_2(j, l, k))\), which is a row vector and a scalar. To further increase efficiency when checking the conditions in Proposition 2, these pairs can be pre-processed and stored. To cover all potentially relevant cases from Proposition 2 only requires the coefficients in \( E_t[x_{t+j}|(l, k)] \) for \( j = \{0, l - 1, l, l + k - 1, l + k\} \), leaving us to only pre-calculate a number of \( l \times k \times 5 \) pairs.

An optimal iterative scheme can be hand-tailored to the problem. For the purpose of this paper, i.e. the estimation of large-scale DSGE models with the ELB on nominal interest rates, we first iterate through \( k = 0, l = 0, 1, \ldots, l_{\text{max}} \). If \( E_t[r_{t+l}|(l, 0)] > \bar{r} \) in each iteration we assert that \((l, k) = (0, 0)\). If this is not the case, we assert that \( k > 0 \) and iterate over all \( k = 1, \ldots, k_{\text{max}} \) with \( l = 0 \) and halt if the conditions in Proposition 1 are satisfied, otherwise increment \( l \) by one and repeat. This iterative scheme is illustrated in Appendix \( A \).

The scheme is very efficient given the specific problem because for the majority of data points used, economic agents to not expect the ELB to be binding and the method will already exit after the first \( l_{\text{max}} \) evaluations. As described below, we use US data from 1964 to 2019, which contain the ELB period from 2008Q4 to 2015Q4. Once the ELB binds, as it does for most post-2008 data points, the transition period is \( l^* = 0 \) and only \( k^* \) needs to be determined. Due to the Federal Reserve’s swift reaction, the policy rate hit the ELB shortly after the outbreak of the Subprime Mortgage crisis such that there are only a few points in time for which both, the transition to the ELB, as well as its expected duration exceed zero periods, and iterations are required for \( l \) and \( k \). While this procedure is tailored to work most efficiently in the context of estimating DSGE models with the ELB, it is generic and applicable to any sort of constraint.

The resulting transition function is linear for the region where the constraint does not bind and nonlinear when it binds.\(^{11}\) Note that this algorithm includes an active assumption on equilibrium selection: if for a \( s_{t-1} \) several sets of \((l^*, k^*)\) exist that satisfy Proposition 2, the set with the lowest \( l^* \) is chosen.

\(^{11}\) Note, that the ELB is the only source of nonlinearity here, i.e. other than e.g. Gust et al. (2017) we do not solve for the global dynamics, and we ignore the effects of economic uncertainty.
3.3 Econometric Methodology

To evaluate the likelihood of a parameter draw we use the Ensemble Kalman Filter (EnKF) introduced in Evensen (1994), which is a hybrid of the particle filter and Kalman filter technology. Although used in many applications ranging from weather forecasting to target tracking, the filter – as pointed out by Katzfuss et al. (2016) – is remarkably unknown in the econometrics community.

Similar to the particle filter, a set of points (the ensemble) is sent through the transition function during the prediction step. However, instead of re-sampling (as with the particle filter), the EnKF approximates a state-dependent system matrix which is used for the Kalman-like updating step. The posterior ensemble of the states, \( X_{t|t} \), is then given by

\[
X_{t|t} = X_{t|t-1} + \bar{X}_{t|t-1} \bar{Z}_{t|t-1}^{-1} \left( \bar{Z}_{t|t-1} \bar{Z}_{t|t-1}^{-1} \right)^{-1} \left( z_t - Z_{t|t-1} \right),
\]

where \( X_{t|t-1} \) is the prior state ensemble, \( Z_{t|t-1} \) is the ensemble of the prior-implied observables, and \( z_t \) the observables vector. Bars on top of the matrices denote deviations from the ensemble mean. The crucial advantage of this filter is that, while it is an approximate full Bayesian filter, it relies on significantly fewer particles than the particle filter. We refer the reader to Appendix B for technical details and focus here on the properties of the filter, and the comparison with alternatives that are currently found in the econometrics literature.

A generic Bayesian filter estimates the sequence of state distributions given potential measurement errors and uncertainty about initial states. A growing econometric literature applies particle filters (PFs, also called Sequential Monte Carlo methods) to economic models and data (see e.g. An and Schorfheide, 2007; Fernández-Villaverde and Rubio-Ramírez, 2007; Herbst and Schorfheide, 2019). The PF requires a high number of particles (i.e. transition function evaluations) that increases fast with the number of state variables. This represents a severe drawback for our application in the context of medium-scale models.\(^\text{12}\) While the PF is an asymptotically unbiased estimator, the EnKF can only be shown to be unbiased in linear systems (Katzfuss et al., 2016), as each prior state distribution – and thereby the inference of the likelihood – is based on a linear approximation based on the prior distribution.\(^\text{13}\) Striktly speaking the EnKF is hence an approximative Bayesian Filter that, in contrast to the PF, allows to efficiently approximate the state distribution of large-scale nonlinear systems with only a few hundred particles.\(^\text{14}\) Nevertheless, for the PF it must be noted that the asymptotic properties likely do not hold for the finite number of particles that is in practice feasible within the scope of estimating the medium-scale models we consider here. Furthermore, the particle filter can be subject to degeneracy issues (see e.g. Binning and Maih, 2015), a problem which is commonly mitigated by assuming counterfactually high measurement errors (MEs). This bears the risk of likelihood misspecification, where the misspecification error involved in PFs grows with the size of the assumed MEs if the true DGP has no or only small MEs (see, Cuba-Borda et al., 2019; Canova et al., 2020). Such sampling degeneracy is not an issue for the EnKF which can generally be used with very small MEs.

\(^{12}\)For the benchmark model of Smets and Wouters (2007) estimates of the number of necessary particles range from at least 40,000 particles in Herbst and Schorfheide (2019) to – more realistically due to the curse of dimensionality – about 1,500,000 particles as in Gust et al. (2017).

\(^{13}\)For linear systems the EnKF gives results identical to the standard Kalman Filter.

\(^{14}\)For all estimations and for the numerical analysis we use ensembles of 400 particles, a number chosen to minimize the standard deviation of the likelihood approximation across random seeds. We sample the initial distribution of states from quasi-random low discrepancy series (e.g. Niederreiter, 1988) to further reduce sampling errors.
As another alternative, Cuba-Borda et al. (2019), drawing on Fair and Taylor (1980), propose an inversion filter (IF) for estimation and filtering of shock innovations. The IF can be understood as solving the nonlinear one-to-one mapping from shocks and observables in each period and approximating the likelihood each period via the probability of the resulting shocks alone. The authors argue that, if initial states are known with certainty and no MEs are specified, the IF is an unbiased estimator of the likelihood. They show that the filter performs well in comparison to the PF for a small-scale model without endogenous states. Similarly, Atkinson et al. (2019) apply the inversion filter in their estimation of small-scale models, assuming no uncertainty about the initial states and no MEs. These restrictions, however, are crucial, since as the IF does not allow for uncertainty about initial states and MEs. As a drawback, bad initial values or moderate jumps in the observables can result in large approximation errors, potentially limiting the filter’s suitability for the estimation of medium-scale models with endogenous states.\footnote{A learning period, as suggested by some authors, will not change this property as, in the absence of potential measurement errors, the course of the dynamics is deterministic. Also note that the filter inherits its numerical properties from the local optimization methods it is based on, which as well can cause instability.}

In contrast to the IF, and like the PF, the EnKF can handle any assumptions regarding the initial state distribution and measurement errors. At the same time it provides the speed that is necessary to handle large models, striking the balance between the PF and the IF. Note that yet another approach would be to directly feed survey data on interest rate expectations into a model augmented by news shocks and forecasting errors (see, e.g., Cai et al., 2019). This is roughly equivalent to feeding a specific series of ELB spell durations into the model, which results in a (conditionally) linear model. Consequently, this procedure ignores any endogenous nonlinearity and the filtered shocks can actually cause different spell durations as the ones initially imposed, which in turn may systematically bias the estimates.\footnote{The discrepancies between simulated spell durations and durations as imposed during estimation are exploited by Jones et al. (2018), who similar to Kulish et al. (2017), include the spell durations in the sampling procedure, which consequently also results in a linear model for each draw. They label such deviations as forward guidance shocks.} Similarly, using a linear Kalman Filter with time-varying coefficients to account for the role of the occasionally binding constraint, would ignore the uncertainty about the distribution of states that comes with the nonlinear transmission of shocks. Since the state-uncertainty is an integral part of Bayesian filters, ignoring it may (and probably will) lead to biased estimates.

Counterfactual simulations require that the smoothened series of shocks can exactly reproduce the filtered data. We utilize a procedure of nonlinear path-adjustment to calculate the historic shock innovations, building two steps on top of the EnKF: the first step is an ensemble version of the Rauch-Tung-Striebel smoother (Rauch et al., 1965; Raanes, 2016). In a second step, iterative global optimization methods are used to maintain that the shock innovations fully respect the nonlinear transition function while taking the approximated distribution of smoothed states into account. Again, Appendix B provides details.

For posterior sampling we apply the differential evolution ensemble Monte Carlo Markov chain method (ter Braak, 2006; ter Braak and Vrugt, 2008, DE-MCMC). DE-MCMC methods are a class of ensemble MCMC methods which, instead of relying on a single or small number of state-dependent chains (as e.g. in the Metropolis algorithm), uses a large number of chains (also called the “ensemble”, now the context of posterior sampling). While the conventional random walk Metropolis algorithm (RWM) generates new proposals using a multivariate normal jump distribu-
tion centered at the current point, the differential evolution algorithm generates new proposals for each chain by adding to the current point a multiple of the difference of the current vectors of two other randomly chosen chains from the ensemble. Thus, the chains “learn” from each other in each updating step, yielding a high speed of convergence. At the same time, the use of many chains ensures a broad search over the parameter space.\textsuperscript{17} Similar ensemble methods have been extensively applied e.g. in astrophysics. The main advantage of these methods is that they are self-tuning, easy to parallelize, and robust against local maxima, which enables them to sample from oddly-shaped and potentially multimodal distributions. Such advanced sampling methods become necessary because models with an endogenous ELB can comprise irregularities for some parameter spaces, such as the reversals discussed in Carlstrom et al. (2015), which can cause additional and large fluctuations of the likelihood even when sampling within a narrow region. For each estimation, we initialize an ensemble of 200 particles with the prior distribution and run 2500 iterations. Of these, we keep 500 as a representation of the posterior distribution. The posterior is hence represented by a sample of $200 \times 500 = 100,000$ parameter vectors. The number of particles is chosen to maintain that our results are reproducible across random seeds.

3.4 Data and Priors

For the quantitative analysis of the Great Recession and its aftermath, we use data from 1964:I to 2019:IV. To our best knowledge, we are the first to include the late 2010’s in the sample, which also contains the exit from the ELB in December 2015. The inclusion of the ELB period in the sample employed in the estimation matters for the model-implied interpretation of the Great Recession. To show this, we additionally consider a pre-crisis sample in our analysis, which extends from 1964:I to 2008:IV.

To allow for a direct comparison of the marginal data densities, we estimate the RANK and the FRANK model on the same set of observables. Those are real GDP growth, real consumption growth, real investment growth, labor hours, the log change of the GDP deflator, real wage growth, the Federal Funds Rate and the BAA spread. For this purpose, we augment the RANK model with an observation equation that links an exogenous AR(1) process directly to the observable spread. Importantly, this exogenous process stands apart from the other model equations such that it does not affect the behaviour of agents in the model.

The measurement equations that relate the model variables to our data series are

\begin{align*}
\text{Real GDP growth} &= \gamma + (y_t - y_{t-1} + z_t), \\
\text{Real consumption growth} &= \gamma + (c_t - c_{t-1} + z_t), \\
\text{Real investment growth} &= \gamma + (i_t - i_{t-1} + z_t), \\
\text{Real wage growth} &= \gamma + (w_t - w_{t-1} + z_t), \\
\text{Labor hours} &= \bar{l} + l_t, \\
\text{Inflation} &= \pi + \pi_t, \tag{32} \\
\text{Federal funds rate} &= \left(\frac{\pi}{\beta_R - \sigma_c} - 1\right) \times 100 + r_t, \tag{33} \\
\text{BAA-spread} &= \text{spread} + \text{spread}_t. \tag{34}
\end{align*}

\textsuperscript{17}ter Braak (2006) provides a well-written introduction into the DE-MCMC and a comprehensive comparison to the conventional RWM.
where in the FRANK model, \( \text{spread}_t \) is defined as \( E_t[r_{t+1}^R - r_t] \) and for RANK it follows an AR(1) process. The construction of the observables is mostly standard and delegated to Appendix D. Consistent with the detrending of nonstationary variables, the growth rate of technology, \( z_t \), in deviations from its steady state enters the measurement equations.

Notably, we set the empirical lower bound of the nominal interest rate within the model to 0.05% quarterly. Setting it exactly to zero would imply that the ELB never binds in our estimations, as the observed series for the FFR stays strictly above zero. Our choice maintains that the ELB is considered binding throughout the period from 2009:Q1 to 2015:Q4. For the observable Federal Funds Rate we cut off any value below 0.05. This maintains that any observable value is also in the domain of model.\(^{18}\)

We assume small measurement errors for all variables with a variance that is 0.01 times the variance of the respective series. Since the Federal Funds rate is perfectly observable (though on a higher frequency) we divide the measurement error variance here again by 100. Hence, the observables are de facto matched perfectly.

In the calibration of some parameters and the choice of the priors for the estimation of the others we stick as closely as possible to the previous literature. For the parameters of RANK we rely on the choices of Smets and Wouters (2007). For the parameters associated with the extension of the financial sector we use the priors employed by Del Negro et al. (2015). In the choice of our prior for \( \tau \), we follow Kulish et al. (2017). Importantly, they opt for a tighter prior for this parameter than Smets and Wouters (2007). Arguably the economy deviated strongly and persistently from its steady state during the Great Recession. In order to dampen the data’s pull of the parameter down to the sample mean, we prefer the tight prior as well.\(^{19}\)

### 4 Business Cycle Dynamics and the Effective Lower Bound

In this section, we use the baseline RANK model to analyze the business cycle dynamics at the ELB. We start by briefly discussing the parameter estimates. We then present the the main implications of the estimated model for the dynamics of the great recession. Finally, we show that the additional post-2008 data points are crucial for the interpretation of the data, and lead to significantly different model dynamics compared to the model estimated on pre-crisis data only.

#### 4.1 Parameter estimates

The summary statistics of the posteriors for the structural parameters for the two main samples are presented in Table 1. We present estimates for the full sample and a pre-crisis sample without the post-crisis data. The latter is comparable to Smets and Wouters (2007).\(^{20}\) As additional information, Table 1 shows estimates for a shorter sample from 1983 to 2019, allowing a closer comparison to previous investigations on the empirical implications of the ELB (see, e.g., Gust et al., 2017; Kulish et al., 2017). While overall, the estimates are well within the range of values previously presented in the literature, there are some crucial differences between the estimates across samples.

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\(^{18}\)The lower bound for the quarterly nominal rate is \( \bar{r} = -100 \left( \frac{\pi}{\beta \gamma - \sigma_c} - 1 \right) + 0.05 \), where \( \pi \) is gross inflation and the parameters \( \gamma \) and \( \sigma_c \) denote the steady state growth rate and the coefficient of relative risk aversion, respectively.

\(^{19}\)For wider priors we confirm unrealistically low estimates of the trend growth rate.

\(^{20}\)Del Negro et al. (2015) re-estimate the model by Smets and Wouters (2007) with data until 2008. Overall, our pre-crisis estimates are similar to those presented in their appendix.
We find that the coefficient of relative risk aversion $\sigma_c$ is slightly above unity in the full sample whereas its mean is higher in the pre-crisis sample (1.5). Similarly, Kulish et al. (2017), who also include the last decade in their estimation, find $\sigma_c$ to be close to unity, similar as in our pre-crisis sample. A value of $\sigma_c$ close to one mutes the effect of variations in labor hours on consumption via the Euler equation, which is introduced through the nonseparabilities in preferences. The reduction of this channel prevents the strong drop in labor hours during the crisis to exert an excessive downwards pull on consumption.

Another difference lies in the estimate of the slope of the Phillips Curve. The pre-crisis estimate of $\zeta_p = 0.714$ is close to the value in the estimation by Smets and Wouters (2007). In contrast, the Calvo parameter of $\zeta_p = 0.904$ in the full sample supports the general notion that the Phillips Curve has flattened in the last decades. This finding is corroborated by estimates of Kulish et al. (2017).

Importantly, the persistence of structural shocks appears to have changed over the last decades. Again, the estimates of these parameters for the pre-crisis sample are well aligned with the results by Smets and Wouters (2007). In contrast, in the full sample, which includes the ELB episode, the risk premium shock display a substantially higher persistence. This points to the increased importance of risk premium shocks in the Great Recession. In turn, the persistence of shock to the marginal efficiency of investment, $\rho_i$ and that of the price markup shock, $\rho_p$ are estimated to be lower in the full sample than in the pre-crisis sample. Lastly, the inclusion of the Great Recession lowers the trend growth rate of the economy, $\gamma$.

Overall, these results are broadly in line with the findings of the previous work that estimates versions of this model while taking the ELB into account as well, for the sample without the ELB period. This lends credence to the results generated with the novel set of methods.

4.2 The Great Recession Through the Lens of RANK

In the context of the RANK model, risk premiums shocks $\epsilon^p_t$ are the most prominent driver of the joint dynamics of key variables following the financial crisis. Figure 1 illustrates the dominant role of this shock for macroeconomic dynamics following the Great Recession.\footnote{The dominant role of risk premium shocks is corroborated by the generalized forecast error variance decomposition. It accounts for roughly half of the variation of output and 60 percent of the variation of the notional rate.} It presents the historical shock decompositions of key variables during the Great Recession based on estimates using the full sample. From 2009 on, persistently elevated risk premiums account for almost the entire drop of aggregate consumption, weigh on aggregate investment and inflation, and consequently are responsible for the long duration of the ELB spell for the nominal interest rate. Christiano et al. (2015) label this shock *consumption wedge* contrasting it with the *financial wedge* that is captured by the MEI shocks in our analysis. Smets and Wouters (2007) compare the effects of the shock to those of disturbances to net worth of entrepreneurs in a model with financial frictions as in Bernanke et al. (1999). Fisher (2015) offers a structural interpretation of the risk premium shock as a shock to the demand for safe and liquid assets. Each of these interpretations share the notion that the risk premium shock is a short cut for capturing some financial disturbances, which makes its prominent role in the Great Recession plausible.

However, high risk premiums cannot fully account for the sharp drop in investment during the Great Recession. While recessionary risk premium shocks do trigger a simultaneous downturn of consumption and investment, they fail to match the drop differential of these components, creating
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the need for an extra driver to make up for the missing decline in investment. In the case at hand, the initial decline of investment is triggered by recessionary MEI shocks, $\epsilon_t^i$, which at the trough account for roughly half of the collapse in investment.

Similarly, the decline of inflation during the Great Recession can only partly be attributed to the increase in risk premiums. The estimated flat Phillips Curve prevents the decline in real activity from generating substantial deflation. It requires price markup shocks, $\epsilon_t^p$, to account for the high-frequency movements of inflation in the sample and account for the dip in inflation during the Great Recession. The only modest decrease in inflation triggered a debate on the missing disinflation puzzle. Christiano et al. (2015) attribute some inflationary pressure to a persistent decline in productivity relative to its pre-recession trend. In contrast, in our estimation, which abstracts from a separate TFP-specific trend, the technology process, $z_t$, is consistently measured to be positive. In addition, Christiano et al. (2015) as well as Gilchrist et al. (2017) ascribe the missing inflation to higher refinancing costs of firms. We cannot confirm within the RANK model that MEI shocks, which increase the firms’ cost of investments, raise inflation. Instead, in our analysis and similar to Del Negro et al. (2015), the estimate of a flat Phillips Curve is responsible
for the lack of a steep decline in inflation. We view the reliance on disparate exogenous drivers for the explanation of the dynamics of key variables at the height of the Great Recession as a failure of the RANK model to ascribe this event to a common source and to provide a joint propagation mechanism.

Figure 2: Estimated expected ELB durations based on the benchmark estimation of RANK. Bars in the top panel mark the mean estimate. The shaded area represents 90% credible sets. The lower panels show histograms of the distribution of ELB durations. The last bar to the right marks the probability of a duration of 10 or more quarters.

The long duration of the ELB is largely interpreted by our estimation as an endogenous response of the central bank to the deterioration of fundamentals via the Taylor rule, rather than to an active lower-for-longer policy.\textsuperscript{22} Figure 2 shows the dynamics and the distribution of the expected duration of the ELB spell over the sample.\textsuperscript{23} The mean expected durations vary between four and ten quarters throughout the ELB years. Although we do not target, nor use any prior information on the actual expectations of market participants on the duration of the ELB, they are broadly comparable to the average expected durations reported by the Blue Chip Financial Forecast and the Federal Reserve Bank of New York’s Survey of Primary Dealers. The lower panels of Figure 2

\textsuperscript{22}In principle, our model allows for forward guidance shocks at the ELB. However, we find that, in the absence of additional data input such as, e.g., term premiums, nonlinear filters do not perform reliably well in identify forward guidance shocks at the ELB. For a discussion of the effects of unconventional monetary policy, see Boehl et al. (2020).

\textsuperscript{23}For a discussion of the economic cost of a binding ELB, see Appendix G. Closely related to the cost of the binding ELB is the decline in the natural rate, which is discussed in Appendix H.
show the distributions of expected ELB durations at different points in time. In 2009:Q1, most of the probability mass lies on a duration of 8 quarters, which is between the 75th and 90th percentile of the distribution implied by survey data. For 2011:Q1, where our mean expected duration of six quarters slightly exceeds the mean implied by the Primary Dealer Survey, the distribution implied by our estimation shows that considerable probability mass is allocated to lower expected durations and the survey mean is within the credible set of the estimation. In the first quarters of 2012 and 2013, for which survey data shows expected durations of ten to eleven quarters, our estimates allots most of the probability mass to seven or six quarters, which is lower than survey data, yet it still implies a substantial role of the ELB.

The resulting estimated average expected durations are higher than those by Gust et al. (2017), who obtain an average ELB spell of merely 3.5 quarters. A potential reason for the difference in the resulting expected durations might be the treatment of the ELB in the estimation. As mentioned in Section 3.4, we set the empirical ELB to 0.05% quarterly, whereas Gust et al. (2017) choose exactly zero percent. This may be problematic as the Federal Funds Rate never actually went all the way down to zero. In theory, their model is hence capable of matching the observables without forcing the model to the zero lower bound.24

Kulish et al. (2017) use the survey data to construct priors on expected durations, which they estimate directly. While this procedure poses a challenge for parameter identification by substantially extending the dimensionality of the parameter space, it eases matching the observed dynamics of the expectations over the years at the ELB. In contrast to the aforementioned papers, our sample also covers the takeoff from the ELB. The mean of the smoothed nominal interest rate series leaves the ELB more than a year after the actual ELB period ended. The model therefore interprets the very low federal fund rate in 2016 and 2017 to have the same effects on equilibrium dynamics as a binding ELB. This might capture uncertainty effects that are not explicitly included in our modeling approach.

4.3 The Merits of Using Post-Crisis Data in the Estimation

Accounting for the ELB in the estimation of a DSGE model is non-trivial (c.f. subsection 3.1). It therefore has become common practice to analyze the dynamics of the US economy during the crisis based on models, that are estimated on pre-ELB data only (see, e.g., Chen et al., 2012; Christiano et al., 2014, 2015; Del Negro et al., 2015; Carlstrom et al., 2017). This approach has generated prominent results that shape our understanding of the Great Recession, the role of financial frictions or the effects of unconventional monetary policy. In this subsection, we illustrate that this practice can yield strongly misleading implications.

Figure 3 shows the historical shock decomposition of key variables in the Great Recession based on the model estimated on the pre-crisis sample without the ELB period. A comparison with Figure 1 highlights one misleading implication that results from the omission of the last decade: the importance of disturbances to the intertemporal investment decision is highly overtaxed. Shocks to investment cost have received heightened attention in a search for an explanation of the events of the Great Recession – Christiano et al. (2015) label it the financial wedge. In their analysis,

24From this angle it is surprising that in their smoothed state estimates, they hit the ELB at all. We suspect that this is due to the assumption of relatively large observation errors, which is often necessary when employing the particle filter (see e.g. Atkinson et al., 2019). Their measurement errors variances are assumed to at least 10% of the variance of data sample, which is a full magnitude higher than our assumed measurement errors (3 magnitudes for the Federal Funds Rate).
variations in this wedge explain the bulk of variations in real activity in the Great Recession and its aftermath. In contrast to their finding, we argue that the importance of disturbances to the intertemporal consumption decision – here risk premium shocks – or in analogous terms, the consumption wedge, is underestimated.\footnote{Fisher (2015) provides an structural interpretation of the shock as a disturbance to the liquidity demand by agents.}

To a good part, this difference in the interpretation of the Great Recession can be traced back to the difference in the estimates of the persistence parameters of risk premium shocks and MEI shocks. Figure 4 illustrates that in the full sample, the effects of risk premium shocks are far more persistent. Additionally, it shows that the fall of investment relative to the decline in consumption in the face of this shock is far less pronounced when the model is estimated on the pre-crisis sample. This is largely due to the difference in the estimates of the coefficient of relative risk aversion, \( \sigma_c \). In the full sample estimate, its posterior mean is close to unity. In the pre-crisis estimate

Figure 3: Historical Shock Decomposition of the Great Recession using the RANK Model estimated on the sample w/o ELB period from 1964–2008. Consumption and Investment: percentage deviations from their steady state growth path. Inflation and (shadow) interest rate: percentage points deviation from steady state. \textit{Note}: Means over 250 simulations drawn from the posterior. The contribution of each shock is normalized as in Appendix C.
it is at 1.5. Already in the full sample estimate, the risk premium shock cannot fully match the drop differential of consumption and investment that was observed in the Great Recession. A risk premium shock that would have triggered a collapse in investment as observed in 2009, would have caused an excessive fall in consumption. For a coefficient of relative risk aversion, as it results from the pre-crisis estimate, this drawback is exacerbated. For values of $\sigma_c$ larger than one, the decline in labor hours exerts an additional downwards pull on consumption through the non-separabilities in the utility function. In turn, the lower consumption translates into an outward shift of the labor supply curve, and a further drop in wages. Investment falls by less, since the marginal product of capital increases with the additional employment used in production. Therefore, the drop differential between investment and consumption becomes even smaller and makes it less likely that risk premium shocks can account for the Great Recession.

In contrast, Figure 5 shows that MEI shocks become more attractive when post-2008 data is omitted from the estimation. In the model estimated on the full sample, a negative MEI shock increases consumption: by lowering aggregate demand, MEI shocks weigh on the policy interest rate, which in turn stimulates consumption on impact. This negative co-movement of consumption and investment is at odds with the observed dynamics in the Great Recession. In the pre-crisis sample, however, both consumption and investment decline with a negative MEI shock. Again, this can be traced back to the difference in the estimate of $\sigma_c$. In the pre-crisis sample, the higher value of $\sigma_c$ strengthens the non-separabilities between labor and consumption. This implies that the decline in labor induces a drop in consumption as well. Notably, the pre-crisis estimate of $\sigma_c$ is very close to the prior mean and it is hard to reject that this estimate is a matter of poor 

Figure 4: Impulse responses to a risk premium shock of one standard deviation in RANK.

Note: Medians over 250 simulations drawn from the posterior. The shaded area depicts the 90% credible set. The shock size equals the posterior mean standard deviation of the shock.
identification. On the contrary, the full sample estimate of this parameter is almost two standard deviations distant from the prior mean, which suggests that the value is driven by the data. Hence we find that through the lens of pre-crisis estimates, MEI shocks – and other financial wedge type of shocks which share similar properties – appear more attractive than they are when including post-2008 data in the estimation.

In summary, the account of the Great Recession offered by our exercise based on the pre-crisis sample differs sharply from the interpretation deemed most likely in the full sample. Here, elevated risk premiums play a dominant role for business cycles. Alternatively to the formalization of Fisher (2015) of these disturbances as shocks to the preference for liquid or safe assets that have an economy-wide impact, positive risk premium shocks can be loosely associated to increases in mortgage lending rates. This calls for a more refined modeling of household finances and additional modeling features that link a contraction in consumption to a strong fall in investment. Apart from the question, which modeling choices prove to be the best fit to capture the events of the recent decade, the exercise in this section highlights the importance of making use of post-2008 data, when analyzing macroeconomic dynamics during this time.

5 On the Role of Financial Frictions

The financial crisis and the following Great Recession triggered an active literature on the role of financial frictions for macroeconomic dynamics. In this section, we discuss the Great Recession through the lens of the extended model, which adds financial frictions a la Bernanke et al. (1999) to
the canonical Smets and Wouters (2007) model. This extension of the model does not improve the performance of the model in explaining the Great Recession. This is due to the fact that the model-implied procyclicality of the leverage in this episode is at odds with the observed countercyclical interest rate spreads. In addition, the decline of the leverage attenuates the effects of shocks to the risk premium and MEI shocks on investment. Thus, the model with financial frictions requires larger shocks to explain the collapse of investment in the Great Recession than the RANK model. We further show that the risk shock à la Christiano et al. (2014), i.e. a disturbance on the financial friction, does not increase the explanatory power of the model.

5.1 Empirical fit: observed spread and model-implied leverage

We estimate the FRANK model on the same set of observables as the RANK model. Table 2 displays the marginal data densities for the baseline model and its extended version with financial frictions. As it turns out, the introduction of financial frictions worsens the empirical performance of the estimated model considerably. The finding of a lower fit is robust to the length of the employed sample. This is startling at first, since the Global Financial crisis and the Great Recession were main motivators for modeling the linkages of the financial sector and the macroeconomy. Indeed, the introduction of financial friction allows for a model-consistent role of financial spreads for business cycle dynamics. As data on spreads are earlier available than national account data, and as they are published on a higher frequency, this provides a substantial advantage for nowcasts and short-term forecasts in a financial crisis. Nonetheless, as we discuss below, this does not imply that the incorporation of financial frictions delivers a better ex-post explanation of the observed macroeconomic series.

The difference between RANK and FRANK is that in the financial frictions model, the required return on capital is linked to the leverage of entrepreneurs (c.f. section 2.2). In addition, the leverage is subject to financial shocks. By construction, a higher leverage increases the default probability of entrepreneurs in the model: to compensate for the increased risk of default, investors charge a higher spread when the leverage is high. This condition is captured by Equation 2, which is repeated here for convenience:

\[
E_t[r_{t+1} - r_t] = u_t + \zeta_{sp,b}(q_t + \bar{k}_t - n_t) + \bar{\sigma}_{w,t}.
\]

This endogenous relation between the leverage ratio and the spread – captured by \(\zeta_{sp,b}(q_t + \bar{k}_t - n_t)\) – is at odds with the implications of the estimated model. Figure 6 shows that the model-implied

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>RANK</td>
<td>-1084.153</td>
<td>-524.8</td>
<td>-331.236</td>
</tr>
<tr>
<td>FRANK</td>
<td>-1177.889</td>
<td>-552.09</td>
<td>-354.075</td>
</tr>
</tbody>
</table>

Table 2: Comparison of Marginal Data Densities

---

26 As discussed in Section 3.4, we augment the RANK model with an extra equation in which an exogenous spread is linked to the observed spread in the estimation.

27 Additionally, the result is also robust to an alternative specification of the observables, in which we group durable consumption into the observed investment series, as in Justiniano et al. (2011).

28 The better forecast performance of the model with financial friction is discussed in Del Negro and Schorfheide (2013); Del Negro et al. (2015); Cai et al. (2019).
leverage ratio of entrepreneurs barely increases 2009, and declines afterwards. At the same time, the
spread, which is directly linked to the observed BAA-spread, remains elevated after its sharp spike
at the height of the recession. The decline of the leverage ratio of entrepreneurs in the model
resembles the observed de-leveraging of financial intermediates in the United States in the years
after 2008 as documented by Adrian and Shin (2014).29

In the estimated model, this deleveraging can be traced back to the substantial decline of the
capital stock, which is depressed by the collapse of investment in the Great Recession. At the same
time, the model generates a price of capital that is somewhat elevated after 2010 while suggesting
that the net worth of entrepreneurs remains largely flat. This divergence between the spread and
its endogenous driver (the leverage ratio) must be compensated by the exogenous risk premium
shocks $u_t$ and the financial shock $\tilde{\sigma}_\omega,t$. Hence, the one additional feature in FRANK that aims at
endogenizing the behavior of the spread fails to explain its dynamics after the Great Recession.
The exogenous shocks necessary to repair this failure and to reconcile the dynamics of the spread
with the dynamics of the observed macroeconomic series worsens the empirical fit of the model.

29More specifically, Adrian and Shin (2014) document in Fig. 5 of their paper that large commercial and investment
banks substantially delevered in the years after the Bear Stearns crisis in March 2008. The authors explain
this observed behavior with the motive to shed risk and to maintain a stable ratio of their Value-at-Risk over their
equity, thereby effectively stabilizing their probability of default.
5.2 The role of the financial sector for the transmission of shocks

The inclusion of the financial sector also alters the transmission of shocks in important ways. The posterior distribution of the parameter estimates for FRANK on the full sample (see table F.5 in Appendix F) shows that the estimated persistence parameters of risk premium shocks as well as of MEI shocks are substantially higher in FRANK than in RANK (table 1). The effects of these shocks are therefore more pronounced and persistent in FRANK than in RANK. Figure 7 illustrates this for the dynamic response of key variables to a positive risk premium shock.

At the same time, however, the risk premium shock in the estimated FRANK model performs worse than the RANK model in generating the relative drop-differential of investment and consumption observed in the Great Recession. On the one hand this is due to the mean estimate of the coefficient of relative risk aversion, which is quite high ($\sigma_c = 1.782$). As discussed in section 4.3, this activates the non-separabilities in the utility function, such that the decline in labor hours in the Great Recession generates excessive downward pressure on aggregate consumption.

Secondly, the financial sector acts as an attenuator of the effects of risk premium shocks on investment. As Figure 7 shows, the risk premium shock increases the spread charged by creditors. However, at the same time it reduces the leverage of entrepreneurs. In response to the shock, the drop in investment lowers the capital stock, but decline of the price of capital and entrepreneurial net worth is rather modest. The subsequent reduction in the leverage compresses the spread that

---

The higher persistence of the effect risk premium shocks in the FRANK model than in the RANK model is in line with previous findings by Cai et al. (2019).
investors demand from entrepreneurs. Hence, the response of the required return on investment is reduced relative to RANK, which in turn lowers the appeal of risk premium shocks for explaining the Great Recession in the FRANK model.

As can be seen in Figure 8, the effects of MEI shocks are altered in the FRANK model as well. While the decline in consumption in response to a recessionary MEI shock constitutes an improvement relative to the estimated RANK model, in which MEI shocks induce a negative co-movement of consumption and investment, the financial sector again acts as an attenuator for investment. More importantly, MEI shocks are supply shocks on the financial markets, which increase the price of capital, while reducing investment and raising entrepreneurial net worth. As a result, this lowers the entrepreneurial leverage and therefore the credit spread. Hence, the shock induces a positive co-movement of investment and the financial spread, which is at odd with the data. This presents a severe drawback of MEI shocks in the FRANK model and rules them out as a candidate for a main driver of joint dynamics of macroeconomic and financial variables in the Great Recession.

5.3 Can risk shocks explain the Great Recession?

An advantage of modeling the financial sector is the possibility to incorporate financial shocks and to study their effect on the real economy. On first sight, this appears to be particularly appealing in the analysis of the Great Recession. The financial shock in the FRANK model is the risk shock, which was developed by Christiano et al. (2014). The risk shock is an exogenous process driving changes in the volatility of cross-sectional idiosyncratic uncertainty of entrepreneurs.
As Figure 9 shows, an increase in entrepreneurial risk raises the credit spread and makes external funding less affordable for entrepreneurs. Aggregate investment and the price of capital therefore both drop, jointly with entrepreneurial net worth. In contrast to the MEI shock, which drives Tobin’s $Q$ and investment in opposite directions, the risk shock is therefore a demand shock in the market for investment goods. The drop in investment demand lowers output and hence labor hours. Given, the non-separabilities in the preferences if households, the decline in labor reduces consumption. With regard of the post-2008 course of inflation, a feature of the risk shock is that, by raising the costs of capital, it increases marginal cost and thereby creates inflationary pressure. However, whereas the risk shock in principle speaks to the missing deflation puzzle, the outright increase in inflation is at odds with observed price dynamics after the recession. As an additional drawback, the higher inflation rate puts upward pressure on the nominal interest rate via the Taylor rule such that its decline after the shock is short-lived. The risk shock therefore cannot explain the drop of the nominal interest towards the zero lower bound.\(^\text{31}\)

Accordingly, the estimated standard deviation of financial shocks is rather small and the effect of the shock on macroeconomic variables is weak compared to, e.g., the effect of a one standard deviation risk premium shock. The historic shock decomposition (c.f. Figure F.10 in Appendix F)

\(^{31}\text{In an estimation of the model on a sample that starts in 1983 and therefore has a narrower focus on the last decades, the risk shock in the spirit of Christiano et al. (2014) is prone to inducing a negative co-movement of investment and consumption. This issue is shared by other financial shocks in the literature (see, e.g., the investment and the credit shock in Carlstrom et al. (2017) or the wealth shock in Carlstrom and Fuerst (1997).}\)
confirms that the role of the financial shock for macroeconomic dynamics is not very prominent. Allowing the risk shock to affect the real economy therefore does not improve upon the explanation of macroeconomic dynamics as given by RANK.

6 Conclusion

This paper applies a novel set of methods to analyze US business cycle dynamics during and after the Great Recession, and decompose the dynamics into the contribution of its causal drivers. It introduces a novel solution method for piece-wise linear models with occasionally binding constraints, whose main advantage over alternative approaches is its computational speed. We complement the solution method with a Bayesian econometric approach, which accounts for the non-linearity induced by the ELB and preserves the method’s advantage in speed. We use these tools to estimate several medium-scale models on a sample that includes the period of the binding ELB as well as the exit from the ELB. With our comprehensive assessment of parameter estimates over various time horizons, we provide reference estimations for this set of models. We show that the inclusion of the Great Recession significantly affects parameter estimates.

We find that post-2008 dynamics are dominated by elevated risk premiums on household borrowing rates, in line with the importance of increased mortgage rates in the financial crisis. In contrast, we find that using pre-crisis-only estimates to analyze the post-2008 period yields the misleading conclusion that shocks to the cost of investment were a main driver for the Great Recession and the US economy’s post-2008 trajectory. This result is a cautionary tale that should discourage from empirically investigating on the Great Recession with models tuned to match the pre-2008 experience.

Importantly, we document that although the empirical performance of the RANK model calls for improvements, the extended model that includes financial frictions as in Bernanke et al. (1999) has a worse empirical fit. This can be traced back to the divergent dynamics of the leverage ratio and the credit spread after the recession. Additional shocks are needed to reconcile the continually elevated spread with the marked delevering in the model. Whereas recessionary financial shocks can in principle be inflationary, their low weight in the estimation prevents them from contributing to an explanation of the missing disinflation puzzle.

Going forward, it is a fruitful endeavor to use more refined models that zoom in on the drivers of elevated risk premiums or to consider a more detailed modeling of labor markets. To keep the scope of the paper manageable, we abstain from a discussion of the role of the expanded set of monetary policies for post-crisis business cycles. Instead, a detailed analysis of the effects of quantitative easing policies for macroeconomic dynamics in the US in the context of a large-scale model is provided by Boehl et al. (2020).
References


Appendix (For Online-Publication)

Appendix A  Details on the solution method

Appendix A.1  Including exogenous shocks

Let the occasionally binding constraint be given by

\[ r_t = \max \{ az_{t+1} + bz_t + cz_{t-1} + d\epsilon_t, \bar{r} \}. \]  \hspace{1cm} (A.1)

The linearized first order conditions of our model can then be represented as

\[ E_t[\mathcal{A}z_{t+1} + \mathcal{B}z_t + \mathcal{C}z_{t-1} + \mathcal{D}\epsilon_t + h \otimes \max \{ az_{t+1} + bz_t + cz_{t-1} + d\epsilon_t, \bar{r} \}] = 0, \]  \hspace{1cm} (A.2)

where \( z_t \) is the \( n_z \)-dimensional vector of all model variables and \( \epsilon_t \) the \( n_\epsilon \)-dimensional vector of iid. exogenous shocks. \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) are generic \( n_z \times n_z \) system matrices whereas, \( \mathcal{D} \) is a \( n_z \times n_\epsilon \) matrix.

Equation (A.2) can elegantly be reduced to

\[ E_t[\hat{A}y_{t+1} + \hat{B}y_t + \hat{C}y_{t-1} + h \otimes \max \{ ay_{t+1} + by_t + cy_{t-1}, \bar{r} \}] = 0, \]  \hspace{1cm} (A.3)

with \( y_t = (z_t, \epsilon_{t+1}) \), \( A = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} \mathcal{B} & 0 \\ 0 & I \end{bmatrix} \), \( C = \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & 0 \end{bmatrix} \) and \( a = (a, 0), b = (b, 0), c = (c, d), h = (h, 0) \), which has the advantage of not having to deal with shocks as an additional term. Define the system in which the constraints is slack (the unconstrained system) as

\[ \hat{A}y_{t+1} + \hat{B}y_t + \hat{C}y_{t-1} = 0, \]  \hspace{1cm} (A.4)

with

\[ \hat{A} = A + h \otimes a, \] \hspace{1cm} (A.5)

\[ \hat{B} = B + h \otimes b, \] \hspace{1cm} (A.6)

\[ \hat{C} = C + h \otimes c. \] \hspace{1cm} (A.7)

In order to maintain stability of the unconstrained system, let us borrow the Assumptions 1 and 2 from Rendahl (2017) and restate them here:

Assumption 1. For any given \( y_0 \in \mathbb{R}^n \), Equation (A.4) has a unique solution, which takes the form \( y_t = Fy_{t-1} \) for \( t \in \mathbb{N}^+ \), where \( F = -(\hat{B} + \hat{A}F)^{-1}\hat{C} \) and where all the eigenvalues of \( F \) are weakly inside the unit circle.

Assume further, to imply that all the eigenvalues of \( F \) are strictly inside the unit circle, that:

Assumption 2. \[ \det \left( \hat{A} + \hat{B} + \hat{C} \right) \neq 0, \] \hspace{1cm} (A.8)

Appendix A.2  Preprocessing and the case of singular \( P \) or \( \hat{P} \)

In the main body of the paper, we assumed for reason of display that \( P \) and \( \hat{P} \) are invertible. Let us now turn to the practically more relevant case where we relax this assumption. Use the QL decomposition on \( M = QL \) and on \( \hat{M} = \hat{Q}\hat{L} \). Let \( n_c \) be the number of control variables. Premultiplication of (11) by \( Q \) (\( \hat{Q} \), respectively), and premultiplication of the \( n_c \) lower rows by the
The inverse of the lower-right \( n_c \times n_c \) submatrix of \( L \) (\( \hat{L} \)) leads to:\textsuperscript{32}

\[
\begin{bmatrix}
\hat{P}_{11} & \hat{P}_{12} \\
\hat{P}_{21} & \hat{P}_{22}
\end{bmatrix} \begin{bmatrix}
s_t \\
E_{ct+1}
\end{bmatrix} = \begin{bmatrix}
\hat{M}_{11} & 0 \\
\hat{M}_{21} & I
\end{bmatrix} \begin{bmatrix}
s_{t-1} \\
c_t
\end{bmatrix} \quad \forall \ p_0 E_t x_{t+1} + m_0 x_t - \bar{r} \geq 0, \quad (A.9)
\]

\[
\begin{bmatrix}
\tilde{P}_{11} & \tilde{P}_{12} \\
\tilde{P}_{21} & \tilde{P}_{22}
\end{bmatrix} \begin{bmatrix}
s_t \\
E_{ct+1}
\end{bmatrix} = \begin{bmatrix}
\tilde{M}_{11} & 0 \\
\tilde{M}_{21} & I
\end{bmatrix} \begin{bmatrix}
s_{t-1} \\
c_t
\end{bmatrix} + \begin{bmatrix}
h_{0,s} \\
h_{0,c}
\end{bmatrix} \bar{r} \quad \forall \ p_0 E_t x_{t+1} + m_0 x_t - \bar{r} < 0. \quad (A.10)
\]

Additionally to the function \( f \) introduced in Equation (14), define \( g \) as

\[
s_t = g(l, k, s_{t-1}), \quad (A.11)
\]

and note that

\[
F_0(l, k, s_{t-1}) = \begin{bmatrix} g(l, k, s_{t-1}) \\ f(l, k, s_{t-1}) \end{bmatrix}. \quad (A.12)
\]

Again, both functions are linear in \( s_{t-1} \) given \( (l, k) \) and take the form

\[
f(l, k, s_{t-1}) = \tilde{f}(l, k)s_{t-1} + \hat{f}(l, k)\bar{r}, \quad (A.13)
\]

\[
g(l, k, s_{t-1}) = \tilde{g}(l, k)s_{t-1} + \hat{g}(l, k)\bar{r}, \quad (A.14)
\]

where \( \hat{f}(0, 0) \) and \( \hat{g}(0, 0) \) are found using any solution routine for linear systems, and \( \tilde{f}(0, 0) = \tilde{g}(0, 0) = \overline{0} \).

For \( k > 0 \) we can then express these functions recursively as

\[
g(l, k, s_{t-1}) = \begin{cases} 
\left( P_{11} + P_{12} \tilde{f}(l - 1, k) \right)^{-1} \left( N_{11} s_{t-1} - P_{12} \hat{f}(l - 1, k) \right) & \text{if } l > 0, \\
\hat{P}_{11} + \hat{P}_{12} \hat{f}(0, k - 1) & \text{if } l = 0.
\end{cases} \quad (A.15)
\]

\[
f(l, k, s_{t-1}) = \begin{cases} 
\left( P_{21} + P_{22} \tilde{f}(l - 1, k) \right) g(l, k, s_{t-1}) - M_{21} s_{t-1} + P_{22} \hat{f}(l - 1, k) & \text{if } l > 0, \\
\tilde{P}_{21} + \tilde{P}_{22} \hat{f}(0, k - 1) & \text{if } l = 0.
\end{cases} \quad (A.16)
\]

As these expressions involve an inversion of a matrix of the same dimensionality of the state space, it is efficient to pre-process all functions within a reasonable range \( 1_{\text{max}} \) and \( k_{\text{max}} \) and store the result for later use. In the same run, \( p_0 E_t [x_{j+1} | (l, k)] + m_0 x_j | (l, k) = p_0 F_{j+1} (l, k, s_{t-1}) + m_0 F_j (l, k, s_{t-1}) \) can be pre-processed and stored for efficient checking of the conditions in proposition 2. This is a \( (1 \times n) \) vector and a scalar for each combination of \( (l, k, s) \) under consideration. Checking each condition in proposition 2 then only requires a dot-vector multiplication and a scalar addition.

Appendix A.3 Iterative scheme

For the purpose of this paper, i.e. the estimation of large-scale DSGE models with the ELB on nominal interest rates, we use the following scheme to find the expected number of periods until

\textsuperscript{32}The fact that the lower-right submatrices of \( L \) and \( \hat{L} \) are nonsingular follows simply from the fact that controls are defined on their future values.
the constraint binds \( l \) and the expected spell duration \( k \):

```python
l, k = 0, 0
for l in range(l_max):
    if b F(l, 0, l, v) - r_bar < 0:
        # constraint binds: interrupt loop
        break
    if l is l_max - 1:
        # return that l=k=0 is an equilibrium
        return 0, 0
... 
```

Hence, if the constraint is not reached within \( l_{\text{max}} \) periods ahead in the future, exit. Otherwise assume \( k > 0 \) and iterate over \( l \) and \( k \) until the equilibrium conditions in (21), (23) and (24) are satisfied:

```python
... 
for l in range(l_max):
    for k in range(1, k_max):
        if l:
            if b F(l, k, 0, v) - r_bar < 0:
                # skip inner loop to next k
                continue
            if b F(l, k, l-1, v) - r_bar < 0:
                # skip inner loop ...
                continue
            if b F(l, k, k+l, v) - r_bar < 0:
                continue
            if b F(l, k, k+l-1, v) - r_bar > 0:
                continue
            if b F(l, k, k+l-1, v) - r_bar > 0:
                continue
            # if we made it here, this must be an equilibrium
            return l, k
# if the loop went though without finding an equilibrium, throw a warning or set error flag
flag = True
warn('No equilibrium exists!11')
```

### Appendix A.4 Benchmarking

Processing and calculation speed are key aspects of the design of the algorithm introduced in Section 3.1. This section presents benchmarks for our method and, additionally, benchmarks for OccBin (c.f. Guerrieri and Iacoviello, 2015). The latter is used frequently in applied work and implemented in Dynare (see Adjemian et al., 2011). For the benchmarks provided below we use the implementation from Cuba-Borda et al. (2019), which goes beyond the standard implementation. In particular, it avoids solving the model and preprocessing the system matrices for every new state.
These steps together are accountable for about 98% of OccBins computation time.\footnote{When benchmarking against the original implementation of Guerrieri and Iacoviello (2015), the method presented in Section 3.1 performs more than 10,000 times faster. For the benchmarks of OccBin, Matlab version R2019a is used with Dynare 4.6.1.}

For each exercise we draw 1,000,000 state vectors from a multivariate normal distribution with zero mean and covariance $\Sigma = 10I_n$, where $n$ is the number of states. Each sample is passed through the nonlinear transition function and grouped according to its calculated expected ZLB duration. We set $l_{\text{max}}=3$ and $k_{\text{max}}=30$ to cover most cases. If within this range no ZLB equilibrium is found, the sample counts as “No ZLB solution”. Note that there are many other reasons why an equilibrium can not be found and a sample may count in this category, e.g. the reversals documented by Carlstrom et al. (2015). For OccBin, $n_{\text{periods}}$ is also set to 30. Although this causes a small number of samples to not converge, it decreases computation time in favor of Occbin. For our method, the overall computation times are 155675 draws per second (6.426 seconds in total). OccBin performed 95.7 draws per second, whereas total computation took 174.13 minutes.

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>std</th>
<th>% of samples</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k^* = 0$</td>
<td>5.435e-06</td>
<td>1.262e-05</td>
<td>48.66%</td>
</tr>
<tr>
<td>$k^* \in (1, 5)$</td>
<td>6.850e-06</td>
<td>4.675e-06</td>
<td>5.79%</td>
</tr>
<tr>
<td>$k^* \in (6, 10)$</td>
<td>9.354e-06</td>
<td>6.214e-06</td>
<td>4.34%</td>
</tr>
<tr>
<td>$k^* \in (11, 15)$</td>
<td>8.323e-06</td>
<td>4.058e-06</td>
<td>8.33%</td>
</tr>
<tr>
<td>$k^* \in (16, 20)$</td>
<td>6.975e-06</td>
<td>1.386e-06</td>
<td>19.99%</td>
</tr>
<tr>
<td>$k^* &gt; 20$</td>
<td>6.894e-06</td>
<td>6.471e-07</td>
<td>12.89%</td>
</tr>
<tr>
<td>$l^* &gt; 0</td>
<td>k^* &gt; 0$</td>
<td>1.410e-05</td>
<td>5.777e-06</td>
</tr>
<tr>
<td>No ZLB solution</td>
<td>3.784e-05</td>
<td>2.307e-06</td>
<td>0.00%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>6.424e-06</td>
<td>9.137e-06</td>
<td>100.00%</td>
</tr>
</tbody>
</table>

Table A.3: Speed benchmark of the method from Section 3.1.

Note: The table shows the execution times per state vector in seconds. “No ZLB solution” collects draws for which no solution was found within a maximum of $l_{\text{max}}$ and $k_{\text{max}}$ periods ahead (two draws in total).

Table A.3 presents the results of the benchmarking exercise for the method from Section 3.1. In about 50% of the samples the ZLB is not binding. For these cases the calculation takes the least time because the algorithm only has to confirm that the ZLB is not binding within the first $l_{\text{max}}$ periods. Calculation time increases with expected ZLB durations, which is an anticipated result given that more guesses are needed. The incremental increase in computation time for higher $k^*$ is, however, rather marginal. Samples for which no solution could be found take almost four times longer than samples in which the ZLB does not bind, which is due to the fact that for these samples all possible combinations of $(l^*, k^*)$ have to be ruled out. However, these are only two samples in total.

The table also documents that the increase in the number of variables reflects less than one-to-one in calculation times. Recall that the first phase of the algorithm – finding $(l^*, k^*)$ – only requires dot multiplications. In vectorized code such calculations generally scale disproportionately relative to the size of the vectors due to the relative reduction of computational fixed costs. The actual execution of $f(l^*, k^*)$ and $g(l^*, k^*)$, ignoring the additive component, has a maximal complexity of $O(n^3)$ and is likely to be faster.
Finally, table A.4 shows speed benchmarks for OccBin. The exercise reveals that our method performs more than 1500 times faster than OccBin. Overall, the percentages of draws in each bin are very similar to the percentages in Table A.3. This confirms that indeed both methods find the same solution, if it is unique. Further, while for the method presented here a draw with an higher $k^*$ does not seem to bear higher computational costs, computation times for OccBin increase with $k^*$. However, the Newton-like method of Occbin seems to require relatively less time to find solutions for draws with $l > 0$, while such draws are relatively more expensive for the method presented here. Lastly, for Occbin 0.12% of all draws no solution is found. While this number is already very low, it can be squeezed down to (almost) zero by setting \texttt{nperiods} to 100.

\section*{Appendix B \ Details on nonlinear filtering}

We here briefly summarize the nonlinear filtering methodology, which is an adaptation of the Ensemble Kalman Filter (Evensen, 1994, EnKF) for the general type of nonlinear problems faced in macroeconomics. Denote a (potentially nonlinear) hidden Markov-Model (HMM) by

\begin{align}
  x_t &= g(x_{t-1}, \varepsilon_t) \\
  z_t &= h(x_t) + \nu_t
\end{align}

with $\varepsilon_t \sim \mathcal{N}(0, Q)$ and $\nu_t \sim \mathcal{N}(0, R)$. Let $X_t = [x_1^t, \ldots, x_N^t] \in \mathbb{R}^{n \times N}$ be the ensemble at time $t$, which consists of $N$ vectors of the state. Further denote by $(\bar{x}_t, P_t)$ the mean and the covariance matrix of the unconditional distribution of states for period $t$. Initialize the ensemble by sampling $N$ times from the prior distribution

\begin{equation}
  X_0 \sim \mathcal{N}(\bar{x}_0, P_0).
\end{equation}
**Step 1: Predict**

Predict the prior-ensemble \( \mathbf{X}_{t-1} \) at time \( t \) by applying the transition function to the posterior ensemble from last period. Use the observation function to obtain a prior-ensemble of observables:

\[
\begin{align*}
\mathbf{X}_{t|t-1} &= \mathbf{g}(\mathbf{X}_{t-1|t-1}, \mathbf{\epsilon}_t), \\
\mathbf{Z}_{t|t-1} &= \mathbf{h}(\mathbf{X}_{t|t-1}) + \mathbf{\nu}_t,
\end{align*}
\] (B.4)

where \( \mathbf{\epsilon}_t \) and \( \mathbf{\nu}_t \) are each \( N \) realizations drawn from the respective distributions.

**Step 2: Update**

Denote by \( \bar{\mathbf{X}}_t = \mathbf{X}_t/(N-1) \) the anomalies of the ensemble, i.e. the deviations from the ensemble mean. Recall that the covariance matrix of the prior distribution at \( t \) is \( \bar{\mathbf{X}}_t \bar{\mathbf{X}}_t^\top (N-1) \). The Kalman mechanism then yields an update-step of

\[
\mathbf{X}_{t|t} = \mathbf{X}_{t|t-1} + \bar{\mathbf{X}}_{t|t-1} \bar{\mathbf{Z}}_{t|1-t}^{-1} \left( \bar{\mathbf{Z}}_{t|1-t} \bar{\mathbf{Z}}_{t|1-t}^\top \right)^{-1} (\mathbf{z}_t - \mathbf{Z}_{t|t-1}).
\] (B.6)

The mechanism is similar to the unscented Kalman filter (UKF), developed by Julier and Uhlmann (1997), but with particles instead of deterministic Sigma points, and statistical linearization instead of the unscented transform. The advantage of the EnKF over the UKF is that its output does not depend on the parametrization of the filter. Conceptually this procedure can be seen as a transposition of the EnKF.\(^{34}\)

The likelihood at each iteration can be then determined by

\[
\mathcal{L}_t = \phi \left( z_t | \bar{\mathbf{X}}_t, \bar{\mathbf{Y}}_t \bar{\mathbf{Y}}_t^\top (N-1) + \mathbf{R} \right).
\] (B.7)

**Appendix B.1 Smoothing and iterative path-adjusting**

For economic analysis we are also interested in the series of shocks, \( \{\mathbf{\epsilon}_t\}_{t=0}^{T-1} \), that fully recovers the mode of the smoothened states. The econometric process of using all available information on all estimates is called smoothing. For this purpose, we employ the Rauch-Tung-Striebel smoother (Rauch et al., 1965) in its Ensemble formulation similar to Raanes (2016).

Denote by \( T \) the period of the last observation available and update each ensemble according to the backwards recursion\(^{35}\)

\[
\mathbf{X}_{t|T} = \mathbf{X}_{t|t} + \bar{\mathbf{X}}_{t|t} \bar{\mathbf{X}}_{t+1|t}^+ \left[ \mathbf{X}_{t+1|T} - \mathbf{X}_{t+1|t} \right].
\] (B.9)

\(^{34}\)Notationally both are equivalent. The regular EnKF assumes the size of the state spaces to be larger than \( N \), and accordingly the term \( \left( \bar{\mathbf{Z}}_{t|1-t} \bar{\mathbf{Z}}_{t|1-t}^\top \right)^{-1} \) to be rank deficient. The mechanism then builds on the properties of the pseudoinverse (the latter provides a least squares solution to a system of linear equations), which is used instead of the regular matrix inverse.

\(^{35}\)Although it is formally correct that

\[
\bar{\mathbf{X}}_{t|t} \bar{\mathbf{X}}_{t+1|t}^+ \left( \bar{\mathbf{X}}_{t+1|T} \bar{\mathbf{X}}_{t+1|T}^\top \right)^{-1} = \bar{\mathbf{X}}_{t|t} \bar{\mathbf{X}}_{t+1|t}^+,
\] (B.8)

the implementation using the LHS of this equation is numerically more stable when using standard implementations of the pseudoinverse based on the SVD.
This creates a series \( \{X_{t|T}\}_{t=0}^T \) of representatives of the distributions of states at each point in time, reflecting all the available information. We now want to ensure that the mode of the distribution fully reflects the nonlinearity of the transition function while retaining a reasonably good approximation of the full distribution. We call this process \textit{nonlinear path-adjustment}. It is important that the smoothened distributions are targeted instead of, e.g., just the distributions of observables and shocks. Only when the full smoothened distributions are targeted it can be maintained that \textit{all} available information from the observables is taken into account. This procedure implicitly assumes that the smoothened distributions approximate the actual transition function sufficiently well and only minor adjustments remain necessary. Since in general there are (many) more states than exogenous shocks, the fitting problem is underdefined and matching precision will depend on the size of the relative (co)variance of each variable. Small observation errors lead to small variances around observable states and tight fitting during path-adjustment while loosely identified states grant more leeway.

Initiate the algorithm with \( \hat{x}_0 = E_{0|T} X \) (the mean vector over the ensemble members), define \( P_{t|T} = \text{Cov}\{X_{t|T}\} \) and for each period \( t \) recursively find

\[
\hat{\varepsilon}_t = \arg \max \{ \log f(\hat{x}_{t-1}, \varepsilon)|\bar{x}_{t|T}, P_{t|T} \},
\]

\[
\hat{x}_t = g(\hat{x}_{t-1}, \hat{\varepsilon}_t),
\]

which can be done using standard iterative methods.

The resulting series of \( \hat{x}_t \) corresponds to the estimated mode given the initial mean and approximated covariances and is completely recoverable by \( \hat{\varepsilon}_t \). Naturally, it represents the nonlinearity of the transition function while taking all available information into account. Since the deviation between mode \( \hat{x}_t \) and mean \( \bar{x}_t \) is in general marginal, we refer to

\[
\{\hat{x}_t, P_t\}_{t=0}^T
\]

as the \textit{path-adjusted smoothed distributions}. \(^{36}\)

### Appendix C Normalization of historic shock decompositions for models with OBCs

We are interested in quantifying the contribution of each type of shock to the time series of the model variables. Such quantification is called the historic shock decomposition (HSD). Once one or several occasionally binding constraints (OBCs) are included in the model, the model is nonlinear and the HSD is generally not unique. To illustrate, imagine a deflationary MEI shock \( \varepsilon^*_t \) and a risk premium shock \( u_t \), which together cause the ELB to bind. Assume that each, the MEI shock and the risk premium shock alone are insufficient to force the ELB to hold. Then, the effect of \( u_t \) conditional on the realization of \( \varepsilon^*_t \) will have a different dynamic effect than just \( u_t \) taken alone, and it is unclear which value to assign to \( u_t \) within a HSD.

\(^{36}\)Unfortunately the adjustment step can not be done during the filtering stage already. Iterative adjustment before the prediction step, would bias the transition of the covariance. Likewise, adjusting after the prediction step will require the repeating the prediction and updating step leading to a potentially infinite loop. See e.g. Ungarala (2012) for details.
More precisely, we are interested in the series of vectors

\[ \{h_{t,z}\}_0^T \]  

where \( z \in \{1, 2, \ldots, n\} \) is in the set of all \( n \) types of shocks. \( \varepsilon_t = (\varepsilon^1_t, \varepsilon^2_t, \ldots, \varepsilon^n_t) \) is the vector of all \( n \) shocks in the model. Each \( h_{t,z} \) is the cumulative dynamic contribution of shock \( z \) to \( v_t \). \( h_{t,z} \) is hence recursive. We require for each period \( t \) that

\[ \sum_{z=1}^n h_{t,z} = v_t, \]  

and at least that

\[ \{h_{t,z} = 0 \land h_{t-1,z} = 0 \iff \varepsilon^z_t = 0\} \forall z = 1, 2, \ldots, n \]  

i.e. that any zero shock has a zero net contribution to the HSD.

We propose a normalization method specific to models with OBCs for historic shock decomposition such that the result is independent of any ordering effects. For convenience, let us repeat Equation (20) from the main body:

\[
F_s(l, k, w_t) = N^{\max\{s-l,0\}} (N + cb)^{\min\{l,s\}} S(l, k, w_t) \\
+ (I - N)^{-1} (I - N^{\max\{s-l,0\}}) c\bar{r}.
\]  

(C.4)

Take as given the time series of smoothed shocks \( \{\varepsilon_t\}_0^T \) that fully reproduces \( \{v_t\}_0^T \). This implies that we also have obtained the series of \( \{l,k\} \). The law-of-motion from period \( t \) to \( t+1 \) is then given by \( F_1(l, k, w_t) \). Note that \( S(l, k, w_t) \) can be decomposed in a coefficient term \( \bar{S}_v(l, k) \), that is to be pre-multiplied to \( w_t \), and a constant term \( \bar{S}_c(k) \) that only depends on \( k \).

Recalling that \( w_t = v_{t-1} + \Xi \varepsilon_t \), we can write

\[
(x_{t+1}, v_t)^\top = \\
F_1(l, k, v_{t-1}, \varepsilon_t) = N^{\max\{1-l,0\}} (N + cb)^{\min\{l+1\}} \bar{S}_v(l, k)v_{t-1} \\
+ N^{\max\{1-l,0\}} (N + cb)^{\min\{l+1\}} \bar{S}_v(l, k)\Xi \varepsilon_t \\
+ N^{\max\{1-l,0\}} (N + cb)^{\min\{l+1\}} \bar{S}_c(k) \\
+ (I - N)^{-1} (I - N^{\max\{1-l,0\}}) c\bar{r}.
\]  

(C.6)

where we are more explicit about the shocks. The first term is linear in \( v_{t-1} \), the second term is linear in \( \varepsilon_t \), whereas the third and forth term are, taking as given \( \{l,k\} \), vectors of constants.

Denote by \( \Xi_z \) the \( z \)-th column of \( \Xi \), which corresponds to the shock \( \varepsilon^z_t \). For each \( z \) we define
\( h_{t,z} \) by the recursion

\[
(f_{t+1,z}, h_{t,z})^T = F_1(l, k, h_{t-1,z}, \Xi_e^z) = N_{\text{max}}(1-l,0) (N + cb)^{\min(l,1)} \tilde{S}_v(l,k)h_{t-1,z} + N_{\text{max}}(1-l,0) (N + cb)^{\min(l,1)} \tilde{S}_v(l,k) \Xi_e^z \]

\[
+ \omega_{t,z} N_{\text{max}}(1-l,0) (N + cb)^{\min(l,1)} \tilde{S}_v(l,k) + \omega_{t,z} (I-N)^{-1}(I-N_{\text{max}}(1-l,0)) c_r,
\]

where it is easy to show that Condition (C.2) is satisfied as long as \( \sum_z \omega_{t,z} = 1 \) \forall t. 37

The first two terms on the RHS of (C.7) are already the recursion of \( h_{t,z} \) and the decomposition respectively. The two other terms are left to be split up and attributed to each shock, which – in terms of (C.7) – implies assigning the weights \( \omega_{t,e} \) such that Condition (C.3) is satisfied.

Define

\[
\omega_{t,z} = \frac{bN_{\text{max}}(1-l,0) (N + cb)^{\min(l,1)} \tilde{S}_v(l,k) (h_{t-1,z} + \Xi_e^z \Xi_e^z)}{bN_{\text{max}}(1-l,0) (N + cb)^{\min(l,1)} \tilde{S}_v(l,k) w_t},
\]

i.e. \( \omega_{t,z} \) is proportional to the relative contribution of \( \varepsilon_e^z \) to the constraint value \( r_t \).

Intuitively, this acknowledges that the values of \( \{l,k\} \) depend on the relation of the scalar \( r_t \) relative to \( \bar{r} \). The further below \( r_t \) is of \( \bar{r} \), the longer the constraint will bind, and the higher is \( k \) (note that the constant term will be zero for any \( l > 0 \)). If the contribution of \( \varepsilon_e^z \) to a negative \( r_t \) is large, then the respective weight \( \omega_{t,z} \) of the constant terms in (C.7) attributed to \( \varepsilon_e^z \) will be high, and vice versa. If however \( h_{t-1,z} \) and \( \varepsilon_e^z \) both are zero, Condition (C.3) is satisfied.

For our application with the ELB this means that the weight of constant terms for each shock is proportional to the shock’s contribution to the total level of the shadow rate. Further note that

\[
\sum_{e} bN_{\text{max}}(1-l,0) (N + cb)^{\min(l,1)} \tilde{S}_v(l,k) (\varepsilon_{t-1,z} = 0) + \varepsilon_{e,t} = bN_{\text{max}}(1-l,0) (N + cb)^{\min(l,1)} \tilde{S}_v(l,k) (\varepsilon_{t-1,z} = 0) + \varepsilon_t,
\]

and hence \( \sum_e \omega_{t,e} = 1 \), i.e. the weights sum up to unity.

**Appendix D Data**

Our measurement equations contain eight variables:

- GDP: \( \ln(GDP/GDPDEF/CNP16OV)^{100} \)
- CONS: \( \ln(PCEC)/GDPDEF/CNP16OV)^{100} \)
- INV: \( \ln(FPI)/GDPDEF/CNP16OV)^{100} \)
- LAB: \( \ln(AWHNONAG*CE16OV)/CNP16OV)^{100} \)
- INFL: \( \ln(GDPDEF) \)

\( f_{t+1,z} \) is a by-product that we do not care about. We want \( h_{t,z} \).
• WAGE: ln(COMPNFB/GDPDEF)*100
• FFR: FEDFUNDS/4
• BAA: (BAAspread)/4

For GDP, CONS, INV, INFL and WAGE we use the log changes in our measurement equations. We demean LAB in our measurement equation.

Data sources:
• GDP: Gross Domestic Product, Billions of Dollars, Quarterly, Seasonally Adjusted Annual Rate, FRED
• GDPDEF: Gross Domestic Product: Implicit Price Deflator, Index 2012=100, Quarterly, Seasonally Adjusted, FRED
• PCEC: Personal Consumption Expenditures, Billions of Dollars, Quarterly, Seasonally Adjusted Annual Rate, FRED
• FPI: Fixed Private Investment, Billions of Dollars, Quarterly, Seasonally Adjusted Annual Rate, FRED
• AWHNONAG: Average Weekly Hours of Production and Nonsupervisory Employees: Total private, Hours, Quarterly, Seasonally Adjusted, FRED.
• CE16OV: Civilian Employment Level, Thousands of Persons, Seasonally Adjusted, FRED.
• CNP16OV: trailing MA(5) of the Civilian Noninstitutional Population, Thousands of Persons, Quarterly, Not Seasonally Adjusted, FRED.
• COMPNFB, Nonfarm Business Sector: Compensation Per Hour, Index 2012=100, Quarterly, Seasonally Adjusted, FRED
• FEDFUNDS: Effective Federal Funds Rate, Percent, FRED.
• BAAspread: BAA Corporate Bond Yield Relative to Yield of 10-Year Treasury Constant Maturity, Percent, Not Seasonally Adjusted, FRED.

Appendix E Model Descriptions

We adopt the framework by Smets and Wouters (2007) as a baseline model to interpret the Great Recession. Following Del Negro and Schorfheide (2013), we detrend all nonstationary variables by

\[ Z_t = e^{\gamma t + \frac{1}{1-\alpha} z_t}, \]

where, \( \gamma \) is the steady-state growth rate of the economy and \( \alpha \) is the output share of capital. \( z_t \) is the linearly detrended log productivity process that follows the autoregressive law of motion

\[ z_t = \rho z_{t-1} + \sigma z_t \epsilon_t. \]

For \( z_t \), the growth rate of technology in deviations from \( \gamma \), it holds that

\[ z_t = \frac{1}{1-\alpha} (\rho z_{t-1} z_t + \frac{1}{1-\alpha} \sigma z_t \epsilon_t). \]

In both models, labor is differentiated by unions with monopoly power that face nominal rigidities for their wage setting process. Intermediate good producers employ labor and capital services and sell their goods to final goods firms. Final good firms are monopolistically competitive and face nominal rigidities as in . The model further allows for exogenous government spending and
features a monetary authority that sets the short-term nominal interest rate according to a monetary policy rule. In FRANK, we assume that frictionless financial intermediates collect funds from households. These funds are lent with a spread, which reflects default risk, to entrepreneurs, who use it together with their own equity to purchase physical capital. Physical capital in turn is rented out to intermediate good producers.

**Appendix E.1 The linearized RANK model**

This subsection briefly presents the linearized equilibrium conditions. A detailed derivation of the linearized equilibrium is discussed e.g. in the appendix to Smets and Wouters (2007). All variables in this section are expressed as a log-deviation from their respective steady state values.

The consumption Euler equation of the households is given by

\[ c_t = \frac{h}{\gamma} (c_{t-1} - z_t) + \frac{1}{1 + h/\gamma} E_t[c_{t+1} + z_{t+1}] + \frac{(\sigma_c - 1)(W^h L/C)}{\sigma_c(1 + h/\gamma)} (l_t - E_t[l_{t+1}]) \]

\[ - \frac{(1 - h/\gamma)}{(1 + h/\gamma)\sigma_c} (r_t - E_t[\pi_{t+1} + u_t]), \tag{E.1} \]

where \( c_t \) is consumption, and \( l_t \) is their supply of labor. Parameters \( h, \sigma_c \) and \( \sigma_l \) are, respectively, the degree of external habit formation in consumption, the coefficient of relative risk aversion, and the inverse of the Frisch elasticity. \( \gamma \) denotes the steady-state growth rate of the economy. \( r_t \) is the nominal interest rate, \( \pi_t \) is the inflation rate, and \( u_t \) is an exogenous risk premium shock, which drives a wedge between the lending/savings rate and the riskless real rate.

Equation (E.2) is the linearized relationship between investment and the relative price of capital,

\[ i_t = \frac{1}{1 + \beta} [(i_{t-1} - z_t) + \frac{\beta}{1 + \beta} E_t[i_{t+1} + z_{t+1}] + \frac{1}{(1 + \beta)^2 \gamma^2 S'' q_t + v_{i,t}}, \tag{E.2} \]

Here, \( i_t \) denotes investment in physical capital and \( q_t \) is the price of capital. It holds that \( \beta = \beta \gamma^{1 - \sigma_c} \) where \( \beta \) is the households’ discount factor. Investment is subject to adjustment costs, which are governed by \( S'' \), the steady-state value of the second derivative of the investment adjustment cost function, and an exogenous process, \( v_{i,t} \). While Smets and Wouters (2007) interpret \( v_{i,t} \) as an investment specific technology disturbance, Justiniano et al. (2011) stress that this shock can as well be viewed as a reduced-form way of capturing financial frictions, as it drives a wedge between aggregate savings and aggregate investment. We henceforth refer to this disturbance as a shock on the marginal efficiency of investment (MEI).

The accumulation equation of physical capital is given by

\[ k_t = (1 - \delta)/\gamma (k_{t-1} - z_t) + (1 - (1 - \delta)/\gamma) i_t + (1 - (1 - \delta)/\gamma)(1 + \beta)\gamma^2 S'' v_{i,t}, \tag{E.3} \]

where \( k \) denotes physical capital, and parameter \( \delta \) is the depreciation rate. The following Equation (E.4) is the no-arbitrage condition between the rental rate of capital, \( r^K \), and the riskless real rate:

\[ r_t - E_t[\pi_{t+1} + u_t] = \frac{r^K}{r^K + (1 - \delta)} E_t[r^K_{t+1}] + \frac{(1 - \delta)}{r^K + (1 - \delta)} E_t[q_{t+1}] - q_t. \tag{E.4} \]

As the use of physical capital in production is subject to utilization costs, which in turn can be expressed as a function of the rental rate on capital, the relation between the effectively used amount
of capital $k_t$ and the physical capital stock is

$$k_t = \frac{1 - \psi}{\psi} r^k_t + k_{t-1},$$  \hspace{1cm} (E.5)$$

where $\psi \in (0, 1)$ is the parameter governing the costs of capital utilization. Equation (E.6) is the aggregate production function

$$y_t = \Phi(\alpha k_t + (1 - \alpha)l_t + z_t) + (\Phi - 1) \frac{1}{1 - \alpha} z_t.$$  \hspace{1cm} (E.6)$$

Intermediate good firms employ labor and capital services. Let $z_t$ be the exogenous process of total factor productivity. Parameter $\alpha$ is the elasticity of output with respect to capital and $\Phi$ enters the production function due to the assumption of a fixed cost in production. Real marginal costs for producing firms, $mc_t$, can be written as

$$mc_t = w_t - z_t + \alpha(l_t - k_t).$$  \hspace{1cm} (E.7)$$

$w_t$ denotes the real wage, which are set by labor unions. Furthermore, cost minimization for intermediate good producers results in condition (E.8):

$$k_t = w_t - r^k_t + l_t.$$  \hspace{1cm} (E.8)$$

The aggregate resource constraint (E.9) contains an exogenous demand shifter, $g_t$, which comprises exogenous variations in government spending and net exports, as well as the resource costs of capital utilization:

$$y_t = G Y g_t + C Y c_t + I Y i_t + R Y K 1 - \psi \frac{1}{\psi} r^k_t + 1 \frac{1}{1 - \alpha} z_t.$$  \hspace{1cm} (E.9)$$

Final good producers are assumed to have monopoly power and face nominal rigidities as in Calvo (1983) when setting their prices. This gives rise to a New Keynesian Phillips Curve (NKPC) of the form

$$\pi_t = \frac{\beta}{1 + \beta_p} E_t \pi_{t+1} + \frac{1_p}{1 + \beta_p} \pi_{t-1} + \frac{(1 - \zeta_p) \beta}{(1 + \beta_p) \zeta_p ((\Phi - 1) \epsilon_p + 1)} mc_t + v_{p,t}.$$  \hspace{1cm} (E.10)$$

Here, $\zeta_p$ is the probability that a firm cannot update its price in any given period. In addition to Calvo pricing, we assume partial price indexation, governed by the parameter $1_p$. The Phillips Curve is hence both, forward and backward looking. $\epsilon_p$ denotes the curvature of the Kimball (1995) aggregator for final goods. Due to the Kimball aggregator, the sensitivity of inflation to fluctuations in marginal cost is affected by the market power of firms, represented by the steady state price markup, $\Phi - 1$.\(^{38}\) Furthermore, the curvature of the Kimball aggregator affects the adjustment of prices to marginal cost as the higher $\epsilon_p$, the higher is the degree of strategic complementarity in price setting, dampening the price adjustment to shocks. The last term in the NKPC, $v_{p,t}$, represents exogenous fluctuations in the price markup.

\(^{38}\)Note that in equilibrium, the steady state price markup is tied to the fixed cost parameter by a zero profit condition.
While final good producers set prices on the good market, wages are set by labor unions. Unions bundle labor services from households and offer them to firms with a markup over the frictionless wage, \( w^h_t \), which reads
\[
 w^h_t = \frac{1}{(1 - h)}(c_t - h/\gamma c_{t-1} + h/\gamma z_t) + \sigma d_t. \tag{E.11}
\]

As with price setting, we assume that the nominal rigidities in the wage setting process are of the Calvo type, and include partial wage indexation. The wage Phillips curve thus is
\[
 w_t = \frac{1}{1 + \beta \gamma}(w_{t-1} - z_t + \pi_{t-1}) + \frac{\beta \gamma}{1 + \beta \gamma} E_t[w_{t+1} + z_{t+1} + \pi_{t+1}] - \frac{1 + \gamma}{1 + \beta \gamma} \pi_t
 + \frac{(1 - \zeta w \beta \gamma)(1 - \zeta w)}{(1 + \beta \gamma)\zeta_w((\lambda_w - 1)\epsilon_w + 1)}(w^h_t - w_t) + v_{w,t}. \tag{E.12}
\]

The term \( w^h_t - w_t \) is the inverse of the wage markup. Analogous to equation (E.10), the terms \( \lambda_w \) and \( \epsilon_w \) are the steady state wage markup and the curvature of the Kimball aggregator for labor services, respectively. The term \( v_{w,t} \) represents exogenous variations in the wage markup.

We take into account the fact that the central bank is constrained in its interest rate policy by a zero lower bound (ELB) on the nominal interest rate. Therefore, in the linear model, it is that
\[
 r_t = \max\{\bar{r}, r^n_t\}, \tag{E.13}
\]
with \( \bar{r} \) being the lower bound value. Whenever the policy rate is away from the constraint, it corresponds to the notational rate, \( r^n_t \), which follows the feedback rule
\[
 r^n_t = \rho r^n_{t-1} + (1 - \rho) (\phi_{\pi} \pi_t + \phi_y \pi_t + \phi_{dy} \Delta y_t + v_{r,t}. \tag{E.14}
\]
Here, \( \bar{y}_t \) is the output gap and \( \Delta \bar{y}_t = \bar{y}_t - \bar{y}_{t-1} \) its growth rate. Parameter \( \rho \) expresses an interest rate smoothing motive by the central bank. \( \phi_{\pi}, \phi_y \) and \( \phi_{dy} \) are feedback coefficients. When the economy is away from the ELB, the stochastic process \( v_{r,t} \) represents a regular interest rate shock. When the nominal interest rate is zero, however, \( v_{r,t} \) may not directly affect the level of the nominal interest rate. However, through the persistence of the stochastic process that drives \( v_{r,t} \), it affects the expected path of the notational rate and can therefore alter the expected duration of the lower bound spell. It can hence be viewed as a forward guidance shock whenever the economy is at the ELB.

The model is augmented with an additional equation to allow for an estimation on the same observables as the FRANK model, including the spread. The equations simply reads
\[
 spread = \bar{\sigma}_{\omega,t},
\]
were the variable ‘spread’ is unrelated to the dynamics of the rest of the model and is driven exclusively by the exogenous shock, \( \bar{\sigma}_{\omega,t} \).
Finally, the stochastic drivers in our model are the following seven processes:

\[ u_t = \rho_u u_{t-1} + \epsilon^u_t, \quad (E.15) \]
\[ z_t = \rho_z z_{t-1} + \epsilon^z_t, \quad (E.16) \]
\[ g_t = \rho_g g_{t-1} + \epsilon^g_t + \rho_{gz} \epsilon^z_t, \quad (E.17) \]
\[ v_{r,t} = \rho_r v_{r,t-1} + \epsilon^r_t, \quad (E.18) \]
\[ v_{i,t} = \rho_i v_{i,t-1} + \epsilon^i_t, \quad (E.19) \]
\[ v_{p,t} = \rho_p v_{p,t-1} + \epsilon^p_t - \mu_p \epsilon^p_{t-1}, \quad (E.20) \]
\[ v_{w,t} = \rho_w v_{w,t-1} + \epsilon^w_t - \mu_w \epsilon^w_{t-1}, \quad (E.21) \]
\[ \tilde{\sigma}_{\omega,t} = \rho_{fin} \tilde{\sigma}_{\omega,t-1} + \epsilon^\omega_t \quad (E.22) \]

where \( \epsilon^k_t \sim iid \sim N(0, \sigma^2_k) \) for all \( k = \{r, i, p, w\} \), and likewise for \( \{u_t, z_t, g_t, \tilde{\sigma}_{\omega,t}\} \).

**Appendix E.2 Financial Frictions**

This subsection lays out the extension of the model: the inclusion of frictions in financial markets. Here, we adopt the modeling choices by Del Negro et al. (2015), who build on the work of Bernanke et al. (1999), and Christiano et al. (2014).

In this model, entrepreneurs obtain loans from frictionless financial intermediates, which in turn receive their funds from household at the riskless interest rate. In addition to the loans, entrepreneurs use their own net worth to finance the purchase of physical capital that they rent out to intermediate good producers. Entrepreneurs are subject to idiosyncratic shocks to their success in managing capital. As a consequence, their revenue might fall short of the amount needed to repay the loan, in which case they will default on their loan. In anticipation of the risk of entrepreneurs’ default, financial intermediates pool their loans and charge a spread on the riskless rate to cover the expected losses arising from defaulting entrepreneurs. Therefore, in the full model, condition (E.4) in the RANK model is replaced by the two conditions

\[ E_{t}[\tilde{r}^k_t - r_t] = u_t + \zeta_{sp,b}(q_t + \bar{K}_t - n_t) + \tilde{\sigma}_{\omega,t}, \quad (E.23) \]
\[ \tilde{r}^k_t - \pi_t = \frac{q_t \bar{r}^k_t}{q_t + (1 - \delta) \bar{r}^k_t} - \frac{q_t + (1 - \delta) \bar{q}_t}{q_t + (1 - \delta) q_t + (1 - \delta) q_{t+1} - q_t - 1}. \quad (E.24) \]

\( \tilde{r}^k_t \) is the nominal return on capital for entrepreneurs, \( n_t \) denotes entrepreneurs’ aggregate net worth, and \( \tilde{\sigma}_{\omega,t} \) allows for exogenous variations in the entrepreneurs’ riskiness. The first condition defines the spread as a function of the entrepreneurs’ leverage and their riskiness, which is determined by the dispersion of the idiosyncratic shocks to entrepreneurs. Note that if the elasticity of the loan rate to the entrepreneurs’ leverage, \( \zeta_{sp,b} \), is set to zero, we are back to the case without financial frictions. Condition (E.24) defines the return on capital for entrepreneurs.

The evolution of aggregate entrepreneurial net worth is described by

\[ n_t = \zeta_{n,\tilde{r}^k}(\tilde{r}^k_t - \pi_t) - \zeta_{n,r}(r_{t-1} - \pi_t) + \zeta_{n,qk}(q_{t-1} + \bar{K}_{t-1}) + \zeta_{n,n}n_{t-1} - \frac{\zeta_{n,\tilde{\sigma}}\tilde{\sigma}_{\omega,t-1} - \gamma_n \nu_{n\omega}}{\zeta_{sp,\tilde{\sigma}}}. \quad (E.25) \]

Equation (E.25) links the accumulated stock of entrepreneurial net worth to the real return of
renting out capital to firms, the riskless real rate, its capital holdings, its past net worth and variations in riskiness. The coefficients $\zeta_n, \tilde{r}_k, \zeta_{n,qk}, \zeta_{n,\sigma_\omega}$, and $\zeta_{sp,\sigma_\omega}$ are derived as in Del Negro et al. (2015). They depend on the steady state calibration of the default rate of entrepreneurs, the distribution of entrepreneurial risk, and their survival probability.

Lastly, the evolution of exogenous variations in entrepreneurial risk, the risk shock in terms of Christiano et al. (2014), follows the process

$$\tilde{\sigma}_{\omega,t} = \rho_\sigma \tilde{\sigma}_{\omega,t-1} + \epsilon_{\sigma,t},$$

(E.26)

with $\epsilon_{\sigma,t} \sim \text{iid} N(0, \sigma^2_\sigma)$. 

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### Appendix F Additional FRANK results

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<th>Prior</th>
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Table F.5: FRANK 1964-2019, 1964-2008
Figure F.10: FRANK Model estimated to 1964-2019. Decomposition of the smoothed time series into the contribution of the different shocks. Output and net worth: percentage deviations from their steady state growth path. Inflation, (shadow) interest rate and spread: percentage points deviation from steady state. Note: Means over 250 simulations drawn from the posterior. The contribution of each shock is normalized as in Appendix C. The shock size is the posterior mean standard deviation for each model.

Appendix G Economic costs of the binding ELB

As we have seen, a negative nominal interest would have been warranted by economic conditions over long parts of the sample. The binding ELB therefore is a constraint that is economically costly. Figure G.11 illustrates these costs for the RANK model estimated on the crisis sample. The bottom panels show that without the ELB, interest rates would have been far in negative territory, with the credible set roughly centered at around -0.4% (1.6% in annual terms) for most of the duration of the ELB spell. We report that this counterfactual stimulus would have hardly increased inflation. However, there would have been economically meaningful gains in output, which would have been up to 1% higher if the ELB would not have been binding. While our results are closely aligned to those reported by Kulish et al. (2017), they stand in contrast to findings by Gust et al. (2017), who, in particular for the Great Recession, report a deeper fall of the notational rate into negative territory. While they report output costs that are roughly similar to ours, the effects of the binding
Figure G.11: RANK model estimated to 1998-2019. On the left: Counterfactual dynamics if the ELB would not have posed a constraint to the nominal rate. On the right: Net effect of the binding ELB.

Note: Means over 250 simulations drawn from the posterior.
ELB on price dynamics are far more pronounced in their framework due to their estimate of a steeper Phillips Curve (0.07 vs. 0.007 in our estimate of the RANK model).

Appendix H  Evolution of the natural rate

Following Laubach and Williams (2003), an active literature has used different approaches to estimate the natural real interest rate, or ‘r-star’. While the most prominent approach is to employ semi-structural models (see, e.g., Laubach and Williams (2003), Holston et al. (2017)), other frameworks such as VARs, VECMs and affine term structure models have been considered in this literature. In addition, Edge et al. (2008) and Neri and Gerali (2019) provide examples for the use of DSGE model in obtaining estimates of the natural rate. As a contribution to this literature, Figure H.12 displays the paths of the US natural rate that are implied by our estimates of the RANK model on several samples. It shows that our model predicts a decline of r-star far into
negative territory after the Financial crisis as well as a return to positive territory at the end of the sample. This finding stands in contrast to estimates of the natural rate according to the models by Laubach and Williams (2003) and Holston et al. (2017), which imply that \( r^* \) remained positive throughout the crisis. Apart from the considerable uncertainty surrounding estimates of \( r^* \), this discrepancy is mainly due to the fact, that DSGE model estimates of the natural rate cannot capture its trend-component. However, according to semi-structural estimates, the trend-growth of output supported \( r^* \) in the financial crisis and kept it in positive territory. In contrast, the path of \( r^* \) in our DSGE model merely captures its cyclical components. Specifically, it reflects fluctuation of the real rate in the frictionless equilibrium around the model’s steady state.

Appendix I The shape of the posterior distribution

The figures in this section show the 200 chains used for the estimation of the benchmark model. We have a total of 2500 samples, of which we keep the last 500. That means that the posterior contains \( 500 \times 200 = 10,000 \) parameter draws. We check for convergence using the method of integrated autocorrelation time (IAT) with a window size of \( c = 50 \), as suggested by Goodman and Weare (2010). Note that it is not trivial to find a sufficient statistics for convergence since the samples in the chain are not independent. In particular, we do not use the plain Ensemble-MCMC sampler that is proposed in Goodman and Weare (2010), but we use the Differential evolution extension as proposed in ter Braak (2006) and ter Braak and Vrugt (2008) as this allows to better handle odd-shaped or bimodal distributions. Naturally, the snooker updater suggested by ter Braak and Vrugt (2008) biases the IAT statistics as it increases autocorrelation times per chain. Independently of these issues concerning the measurement of convergence, the figures strongly suggest that the estimation is converged from iteration 1000 onwards.

Figure I.13: Traceplots of the 200 DE-MCMC chains for selected parameters. Estimation of the benchmark model. The left panel shows a KDE of the parameter distribution. The right displays the trace of each of the chains over time.
Figure I.14: Tranceplots of the 200 DE-MCMC chains for selected parameters. Estimation of the benchmark model. The left panel shows a KDE of the parameter distribution. The right displays the trace of each of the chains over time.

Figure I.15: Tranceplots of the 200 DE-MCMC chains for selected parameters. Estimation of the benchmark model. The left panel shows a KDE of the parameter distribution. The right displays the trace of each of the chains over time.
Figure I.16: Tranceplots of the 200 DE-MCMC chains for selected parameters. Estimation of the benchmark model. The left panel shows a KDE of the parameter distribution. The right displays the trace of each of the chains over time.

Figure I.17: Tranceplots of the 200 DE-MCMC chains for selected parameters. Estimation of the benchmark model. The left panel shows a KDE of the parameter distribution. The right displays the trace of each of the chains over time.
Figure I.18: Tranceplots of the 200 DE-MCMC chains for selected parameters. Estimation of the benchmark model. The left panel shows a KDE of the parameter distribution. The right displays the trace of each of the chains over time.

Figure I.19: Tranceplots of the 200 DE-MCMC chains for selected parameters. Estimation of the benchmark model. The left panel shows a KDE of the parameter distribution. The right displays the trace of each of the chains over time.